Neutrosophic Operational Research

Volume I

Editors:

Prof. Florentin Smarandache

Dr. Mohamed Abdel-Basset

Dr. Yongquan Zhou

NEUTROSOPHIC OPERATIONAL RESEARCH

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Dedication

Dedicated with love to our parents for the developments of our cognitive minds, ethical standards, and shared do-good values & to our beloved families for the continuous encouragement, love, and support.

Acknowledgment

The book would not have been possible without the support of many people: first, the editors would like to express their appreciation to the advisory board; second, we are very grateful to the contributors; and third, the reviewers for their tremendous time, effort, and service to critically review the various chapters. The help of top leaders of public and private organizations inspired, encouraged, and supported the development of this book.

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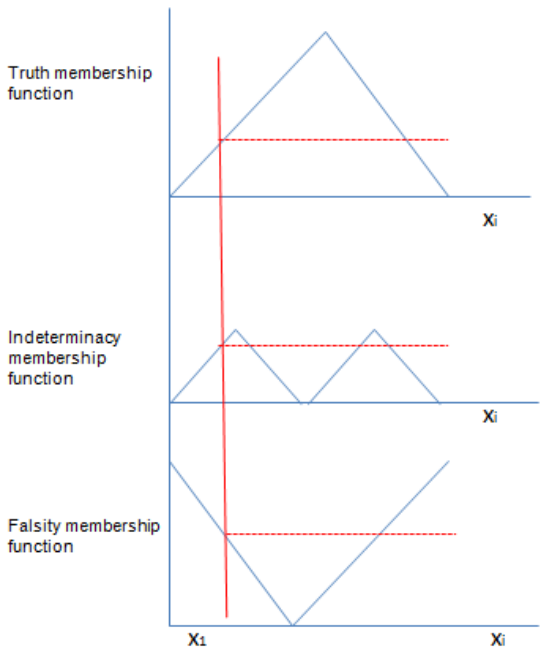
Volume I

Foreword by John R. Edwards

Preface by the editors



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Foreword

John R. Edwards

This book is an excellent exposition of the use of Data Envelopment Analysis (DEA) to generate data analytic insights to make evidence-based decisions, to improve productivity, and to manage cost-risk and benefit-opportunity in public and private sectors. The design and the content of the book make it an up-to-date and timely reference for professionals, academics, students, and employees, in particular those involved in strategic and operational decision-making processes to evaluate and prioritize alternatives to boost productivity growth, to optimize the efficiency of resource utilization, and to maximize the effectiveness of outputs and impacts to stakeholders. It is concerned with the alleviation of world changes, including changing demographics, accelerating globalization, rising environmental concerns, evolving societal relationships, growing ethical and governance concern, expanding the impact of technology; some of these changes have impacted negatively the economic growth of private firms, governments, communities, and the whole society.

Preface

Prof. Florentin Smarandache

Dr. Mohamed Abdel-Basset

Dr. Yongquan Zhou

This book treats all kind of data in neutrosophic environment, with real-life applications, approaching topics as linear programming problem, linear fractional programming, integer programming, triangular neutrosophic numbers, single valued triangular neutrosophic number, neutrosophic optimization, goal programming problem, Taylor series, multi-objective programming problem, neutrosophic geometric programming, neutrosophic topology, neutrosophic open set, neutrosophic semi-open set, neutrosophic continuous function, cylindrical skin plate design, neutrosophic MULTIMOORA, alternative solutions, decision matrix, ratio system, reference point method, full multiplicative form, ordinal dominance, standard error, market research, and so on.

The first chapter (*Neutrosophic Linear Programming Problems*) proposes some linear programming problems based on neutrosophic environment. Neutrosophic sets are characterized by three independent parameters, namely truth-membership degree (*T*), indeterminacy-membership degree (*I*) and falsity-membership degree (*F*) which are more capable to handle imprecise parameters. The authors (Abdel-Nasser Hussian, Mai Mohamed, Mohamed Abdel-Baset, and Florentin Smarandache) also transform the neutrosophic linear programming problem into a crisp programming model by using neutrosophic set parameters. To measure the efficiency of the proposed model, several numerical examples are solved.

A solution procedure is proposed in the second chapter (*Neutrosophic Linear Fractional Programming Problems*), to solve neutrosophic linear fractional programming (NLFP) problem where cost of the objective function, the resources and the technological coefficients are triangular neutrosophic numbers. The NLFP problem is transformed into an equivalent crisp multi-objective linear fractional programming (MOLFP) problem. By using the proposed approach, the transformed MOLFP problem is reduced to a single objective linear programming (LP) problem which can be solved easily by suitable LP problem algorithm. The proposed procedure is illustrated through a

numerical example by authors Abdel-Nasser Hussian, Mai Mohamed, Mohamed Abdel-Baset, Yongquan Zhou, Florentin Smarandache.

In the third chapter (*Neutrosophic Integer Programming Problems*), Mohamed Abdel-Baset, Mai Mohamed, Abdel-Nasser Hessian, and Florentin Smarandache introduce the integer programming in neutrosophic environment, by considering coefficients of problem as a triangular neutrosophic numbers. The degrees of acceptance, indeterminacy and rejection of objectives are simultaneously considered. The Neutrosophic Integer Programming Problem (NIP) is transformed into a crisp programming model, using truth membership (T), indeterminacy membership (I), and falsity membership (F) functions as well as single valued triangular neutrosophic numbers. To measure the efficiency of the proposed model, several numerical examples are solved.

Ibrahim M. Hezam, Mohamed Abdel-Baset, and Florentin Smarandache introduce in the fourth chapter (*Neutrosophic Goal Programming*) two models to solve Neutrosophic Goal Programming Problem (NGPP), with the goal to minimize the sum of the deviation in the model (I), while in the model (II), neutrosophic goal programming problem NGPP is transformed into the crisp programming model using truth membership, indeterminacy membership, and falsity membership functions. An industrial design problem is given to illustrate the efficiency of the proposed models. The obtained results of model (I) and model (II) are compared with other methods.

In the fifth chapter (*Taylor Series Approximation to Solve Neutrosophic Multiobjective Programming Problem*), Taylor series is used to solve neutrosophic multi-objective programming problem (NMOPP). In the proposed approach, the truth membership, indeterminacy membership, falsity membership functions associated with each objective of multi-objective programming problems are transformed into a single objective linear programming problem by using a first order Taylor polynomial series. To illustrate the efficiency of the proposed method, a numerical experiment for supplier selection is given as an application of Taylor series method for solving neutrosophic multi-objective programming problem. The authors of this research are Ibrahim M. Hezam, Mohamed Abdel-Baset, and Florentin Smarandache.

Mridula Sarkar and Tapan Kumar Roy develop in the sixth chapter (*Multi-objective Cylindrical Skin Plate Design Optimization based on Neutrosophic Optimization Technique*), a Neutrosophic Optimization (NSO) approach for optimizing the thickness and sag of skin plate of vertical lift gate with multi-objective subject to a specified constraint. In this optimum design formulation, the objective function is the thickness and sag of the skin plate of vertical lift gate; the design variables are the thickness and sag of skin plate of vertical lift gate;

the constraint are the stress and deflection in member. A classical vertical lift gate optimization example is presented to demonstrate the efficiency of this technique. The test problem includes skin plate of vertical lift gate subjected to hydraulic load condition. This multi-objective structural optimization model is solved by fuzzy, intuitionistic fuzzy and neutrosophic multi-objective optimization technique. A numerical example is given to illustrate the NSO approach. The result shows that the NSO approach is very efficient in finding the best discovered optimal solutions.

The siventh chapter, called *Neutrosophic MULTIMOORA: A Solution for the Standard Error in Information Sampling*, is authored by Willem K. M. Brauers, Alvydas Baležentis, and Tomas Baležentis. If complete Data Mining is not possible, one has to be satisfied with an information sample, as representative as possible. How and when Multi-Objective Optimization Methods is helpful? The researchers take an example. The Belgian company "CIM" is doing marketing research for all Belgian newspapers, magazines, and cinema. For some local newspapers, it arrives at a standard error of more than 15% or a spread of more than 30%, which is scientific nonsense, but accepted by the publishers of advertisement. On the other side, technical problems will ask for a much smaller standard deviation like for instance a standard error of 0.1% for the possibility that a dike is not strong enough for an eventual spring tide. Somewhat in between the usual standard error for marketing research is 5%. Is it possible to avoid this Spread by Sampling? The Neutrosophic MULTIMOORA method, chosen for its robustness compared to many other competing methods, will solve the problems of normalization and of importance, whereas Fuzzy MULTIMOORA may take care of the annoying spread in the marketing samples. While an application on the construction of dwellings is given, many other applications remain possible, e.g. for Gallup polls concerning public opinion, or general elections in particular.

The eight chapter (*Multi-objective Geometric Programming Problem Based on Neutrosophic Geometric Programming Technique*) aims to give computational algorithm to solve a multi-objective non-linear programming problem using Neutrosophic geometric programming technique. As the Neutrosophic optimization technique utilizes degree of truth-membership, falsity-membership and indeterminacy-membership functions, the authors Pintu Das and Tapan Kumar Roy made a study of correspondence among those membership functions to see its impact on optimization. Also, they made a comparative study of optimal solution between intuitionistic fuzzy geometric programming and Neutrosophic geometric programming technique. The developed algorithm has been illustrated by a numerical example. An application of proposed Neutrosophic geometric programming technique on gravel box design problem is presented.

The ninth chapter (*Using Neutrosophic Sets to Obtain PERT Three-Times Estimates in Project Management*) attempts to introduce the mathematical representation of Program Evaluation and Review Technique (PERT) in neutrosophic environment. Here the elements of three-times estimates of PERT are considered as neutrosophic elements. Score and accuracy functions are used to obtain crisp model of problem. The proposed method has been demonstrated by a suitable numerical example by the authors Mai Mohamed, Mohamed Abdel-Basset, Abdel-Nasser Hussien, and Florentin Smarandache.

The Critical Path Method (CPM) is one of several related techniques for planning and managing of complicated projects in real world applications. In many situations, the data obtained for decision makers are only approximate, which gives rise of neutrosophic critical path problem. In the tenth chapter (*A Critical Path Problem Using Triangular Neutrosophic Number*), the proposed method has been made to find the critical path in network diagram. The vague parameters in the network are represented by triangular neutrosophic numbers, instead of crisp numbers. Two illustrative examples are provided to validate the proposed approach by authors Mai Mohamed, Mohamed Abdel-Baset, Florentin Smarandache, Yongquan Zhou.

The same researchers discuss in the eleventh chapter (*A Critical Path Problem in Neutrosophic Environment*) about a mathematical model of neutrosophic CPM and propose an algorithm. Project management is concerned with selecting, planning, execution and control of projects in order to meet or exceed stakeholders' need or expectation from project. Two techniques of project management, namely Critical Path Method (CPM) and Program Evaluation and Review Technique (PERT) where developed in 1950s. The successful implementation of CPM requires clear determined time duration for each activity. Steps involved in CPM include: Develop Work Breakdown Structure of a project, estimate the resources needed and establish precedence relationship among activities; Translate the activities into network; Carry out network computation and prepare schedule of the activities. In CPM, the main problem is wrongly calculated activity durations, of large projects that have many activities. The planned value of activity duration time may change under certain circumstances and may not be presented in a precise manner due to the error of the measuring technique or instruments etc. It has been obvious that neutrosophic set theory is more appropriate to model uncertainty that is associated with parameters such as activity duration time and resource availability in CPM.

Uncertainty and indeterminacy are two major problems in data analysis these days. Neutrosophy is a generalization of the fuzzy theory. Neutrosophic system is based on indeterminism and falsity of concepts in addition to truth

degrees. Any neutrosophy variable or concept is defined by membership, indeterminacy and non-membership functions. Finding efficient and accurate definition for neutrosophic variables is a challenging process. The next chapter (*Integrated Framework of Optimization Technique and Information Theory Measures for Modeling Neutrosophic Variables*) presents a framework of Ant Colony Optimization and entropy theory to define a neutrosophic variable from concrete data. Ant Colony Optimization is an efficient search algorithm presented to define parameters of membership, indeterminacy and non-membership functions. The integrated framework of information theory measures and Ant Colony Optimization is proposed. Experimental results contain graphical representation of the membership, indeterminacy and non-membership functions for the temperature variable of the forest fires data set. The graphs demonstrate the effectiveness of the proposed framework.

In Geographical information systems (GIS) there is a need to model spatial regions with indeterminate boundary and under indeterminacy. The purpose of the final chapter (*New Neutrosophic Sets via Neutrosophic Topological Spaces*) is to construct the basic concepts of the so-called "neutrosophic sets via neutrosophic topological spaces (NTs)". After giving the fundamental definitions and the necessary examples, we introduce the definitions of neutrosophic open sets, neutrosophic continuity, and obtain several preservation properties and some characterizations concerning neutrosophic mapping and neutrosophic connectedness. Possible applications to GIS topological rules are touched upon.

I

Neutrosophic Linear Programming Problems

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Abstract

Smarandache presented neutrosophic theory as a tool for handling undetermined information. Wang et al. introduced a single valued neutrosophic set that is a special neutrosophic sets and can be used expediently to deal with real-world problems, especially in decision support. In this paper, we propose linear programming problems based on neutrosophic environment. Neutrosophic sets are characterized by three independent parameters, namely truth-membership degree (T), indeterminacy-membership degree (I) and falsity-membership degree (F), which are more capable to handle imprecise parameters. We also transform the neutrosophic linear programming problem into a crisp programming model by using neutrosophic set parameters. To measure the efficiency of our proposed model we solved several numerical examples.

Keywords

Linear Programming Problem; Neutrosophic; Neutrosophic Sets.

1 Introduction

Linear programming is a method for achieving the best outcome (such as maximum profit or minimum cost) in a mathematical model represented by linear relationships. Decision making is a process of solving the problem and achieving goals under asset of constraints, and it is very difficult in some cases due to

incomplete and imprecise information. And in Linear programming problems the decision maker may not be able to specify the objective function and/or constraints functions precisely. In 1995, Smarandache [5-7] introduce neutrosophy which is the study of neutralities as an extension of dialectics. Neutrosophic is the derivative of neutrosophy and it includes neutrosophic set, neutrosophic probability, neutrosophic statistics and neutrosophic logic. Neutrosophic theory means neutrosophy applied in many fields of sciences, in order to solve problems related to indeterminacy. Although intuitionistic fuzzy sets can only handle incomplete information not indeterminate, the neutrosophic set can handle both incomplete and indeterminate information. [2,5-7] Neutrosophic sets characterized by three independent degrees namely truth-membership degree (T), indeterminacy-membership degree(I), and falsity-membership degree (F), where T,I,F are standard or non-standard subsets of $]0^-,1^+[$. The decision makers in neutrosophic set want to increase the degree of truth-membership and decrease the degree of indeterminacy and falsity membership.

The structure of the paper is as follows: the next section is a preliminary discussion; the third section describes the formulation of linear programing problem using the proposed model; the fourth section presents some illustrative examples to put on view how the approach can be applied; the last section summarizes the conclusions and gives an outlook for future research.

2 Some Preliminaries

2.1 Neutrosophic Set [2]

Let X be a space of points (objects) and $x \in X$. A neutrosophic set A in X is defined by a truth-membership function T (x), an indeterminacy-membership function (x) and a falsity-membership function (x). $T(x)$, $I_A(x)$ and $F_A(x)$ are real standard or real nonstandard subsets of $]0^-,1^+[$. That is $T_A(x):X \rightarrow]0^-,1^+[$, $I_A(x):X \rightarrow]0-,1+[$ and $F_A(x):X \rightarrow]0^-,1^+[$. There is no restriction on the sum of $T(x)$, $I_A(x)$ and $F_A(x)$, so $0^- \leq T_A(x) \leq \sup I_A(x) \leq F_A(x) \leq 3^+$.

2.2 Single Valued Neutrosophic Sets (SVNS) [7,8]

Let X be a universe of discourse. A single valued neutrosophic set A over X is an object having the form

$A = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle : x \in X\}$, where $T_A(x):X \rightarrow [0,1]$, $I_A(x):X \rightarrow [0,1]$ and $F_A(x):X \rightarrow [0,1]$ with $0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$ for all $x \in X$. The intervals $T(x)$, $I_A(x)$and $F_A(x)$ denote the truth-membership degree, the indeterminacy-membership degree and the falsity membership degree of x to A, respectively.

For convenience, a SVN number is denoted by A= (a, b, c), where a, b, c ∈ [0, 1] and a+b+c≤3.

2.3 Complement [3]

The complement of a single valued neutrosophic set A is denoted by $_C$ (A) and is defined by

$T_{c(A)(x)} = F(A)(x)$,

$I_c(A)(x) = 1 - I(A)(x)$,

$F_c(A)(x) = T(A)(x)$, for all x in X.

2.4 Union [3]

The union of two single valued neutrosophic sets A and B is a single valued neutrosophic set C, written as C = A∪B, whose truth-membership, indeterminacy membership and falsity-membership functions are given by

$T(C)(x) = max\ (\ T(A)(x), T(B)(x)\ )$,

$I(C)(x) = max\ (I(A)(x), I(B)(x))$,

$F(C)(x) = \ min(\ F(A)(x), F(B)(x)\ )$ for all x in X.

2.5 Intersection [3]

The intersection of two single valued neutrosophic sets A and B is a single valued neutrosophic set C, written as C = A∩B, whose truth-membership, indeterminacy membership and falsity-membership functions are given by

$T(C)(x) = min\ (\ T(A)(x)\ ,T(B)(x)\ )$,

$I(C)(x) = min\ (\ (A)(x)\ ,I(B)(x)\ )$, $F(C)(x) = max(\ F(A)(x)\ ,F(B)(x)\ )$ for *all* x *in* X

3 Neutrosophic Linear Programming Problem

Linear programming problem with neutrosophic coefficients (NLPP) is defined as the following:

Maximize Z= $\sum_{j=1}^{n} c_j x_j$

Subject to

$\sum_{j=1}^{n} a_{ij}^{\sim n} x_j \le b_i \quad 1 \le i \le m$ (1)

$x_j \ge 0, \qquad 1 \le j \le n$

where a_{ij}^{n} *is* a neutrosophic number.

The single valued neutrosophic number (a_{ij}^n) is donated by A=(a,b,c) where a,b,c ∈ [0,1] And a,b,c ≤ 3

The truth- membership function of neutrosophic number a_{ij}^n is defined as:

$$T\,a_{ij}^n(x) = \begin{cases} \frac{x-a_1}{a_2-a_1} & a_1 \le x \le a_2 \\ \frac{a_2-x}{a_{3-a_2}} & a_2 \le x \le a_3 \\ 0 \quad otherwise \end{cases} \tag{2}$$

The indeterminacy- membership function of neutrosophic number $a_{ij}^n$is defined as:

$$I\,a_{ij}^n(x) = \begin{cases} \frac{x-b_1}{b_2-b_1} & b_1 \le x \le b_2 \\ \frac{b_2-x}{b_{3-b_2}} & b_2 \le x \le b_3 \\ 0 \;\; otherwise \end{cases} \tag{3}$$

And its falsity- membership function of neutrosophic number $a_{ij}^{\sim n}$ is defined as:

$$F\,a_{ij}^n(x) = \begin{cases} \frac{x-C_1}{C_2-C_1} & C_1 \le x \le C_2 \\ \frac{C_2-x}{C_{3-C_2}} & C_2 \le x \le C_3 \\ 1 \;\; otherwise \end{cases} \tag{4}$$

Then we find the upper and lower bounds of the objective function for truth-membership, indeterminacy and falsity membership as follows:

$z_U^T = \max\{z(x_i^*)\}$ and z_l^T =min$\{z(x_i^*)\}$ where 1≤ $i \le k$

$z_{L=}^F z_L^T$ And $z_{u=}^F z_u^T - R(z_u^T - z_L^T)$

$z_{U=}^I z_U^I \; and \; z_{l=}^I z_{l=}^I - S(z_u^T - z_L^T)$

where R, S are predetermined real number in (0, 1).

The truth membership, indeterminacy membership, falsity membership of objective function are as follows:

$$T_O^{(Z)} = \begin{cases} 1 & if \;\; z \ge z_u^T \\ \frac{z-z_L^T}{z_u^T-z_L^T} & if \;\; z_L^T \le z \le z_u^T \\ 0 & if \; z \; < z_L^T \end{cases} \tag{5}$$

$$I_O^{(Z)} = \begin{cases} 1 & if \; z \ge z_u^T \\ \frac{z-z_L^I}{z_u^I-z_L^I} & if \;\; z_L^T \le z \le z_u^T \\ 0 & if \; z \; < z_L^T \end{cases} \tag{6}$$

$$F_O^{(Z)} = \begin{cases} 1 & if\ z \geq z_u^T \\ \frac{z_u^F - Z}{z_u^F - z_L^F} & if\ \ z_L^T \leq z \leq z_u^T \\ 0 & if\ z\ < z_L^T \end{cases} \quad (7)$$

The neutrosophic set of the i^{th} constraint c_i is defined as:

$$T_{c_i}^{(x)} = \begin{cases} 1 & if \quad b_i \geq \sum_{j=1}^{n}(a_{ij} + d_{ij})x_j \\ \frac{b_i - \sum_{j=1}^{n} a_{ij}x_j}{\sum_{j=1}^{n} d_{ij}x_j} & if \quad \sum_{j=1}^{n} a_{ij}x_j \leq b_i < \sum_{j=1}^{n}(a_{ij} + d_{ij})x_j \\ 0 & if \quad b_i\ < \sum_{j=1}^{n} a_{ij}x_j \end{cases} \quad (8)$$

$$I_{c_i}^{(x)} = \begin{cases} 0 \ \ if \quad b_i < \sum_{j=1}^{n} a_{ij}x_j \\ \frac{b_i - \sum_{j=1}^{n} d_{ij}x_j}{\sum_{j=1}^{n} a_{ij}x_j}\ If \quad \sum_{j=1}^{n} a_{ij}x_j \leq b_i < \sum_{j=1}^{n}(a_{ij} + d_{ij})x_j \\ 0\ if\ b_i \geq \sum_{j=1}^{n}(a_{ij} + d_{ij})x_j \end{cases} \quad (9)$$

$$F_{c_i}^{(x)} = \begin{cases} 1 & if \quad b_i < \sum_{j=1}^{n} a_{ij}x_j \\ 1 - T_{c_i}^{(x)} & if \quad \sum_{j=1}^{n} a_{ij}x_j \leq b_i < \sum_{j=1}^{n}(a_{ij} + d_{ij})x_j \\ 0 & if\ b_i \geq \sum_{j=1}^{n}(a_{ij} + d_{ij})x_j \end{cases} \quad (10)$$

4 Neutrosophic Optimization Model

In our neutrosophic model we want to maximize the degree of acceptance and minimize the degree of rejection and indeterminacy of the neutrosophic objective function and constraints. Neutrosophic optimization model can be defined as:

$$maxT_{(x)}$$

$$minF_{(x)}$$

$$minI_{(x)}$$

Subject to

$$T_{(X)} \geq F_{(x)}\ ,$$

$$T_{(X)} \geq I_{(x)}\ ,$$

$$0 \leq T_{(X)} + I_{(x)} + F_{(x)} \leq 3, \quad (11)$$

$$T_{(X)},\quad I_{(X)}\ ,\quad F_{(X)} \geq 0,\quad x \geq 0\ ,$$

where $T_{(x)}, F_{(x)}, I_{(x)}$ denote the degree of acceptance, rejection, and indeterminacy of x respectively.

The above problem is equivalent to the following:

$max\ \alpha,\ min\ \beta\ , min\ \theta$

Subject to

$$\alpha \leq \mathrm{T}\ (x),$$
$$\beta \leq \mathrm{F}\ (x),$$
$$\theta \leq \mathrm{I}\ (x),$$
$$\alpha \geq \beta,$$
$$\alpha \geq \theta,$$
$$0 \leq\ \alpha + \beta + \theta \leq 3, \qquad (12)$$
$$x \geq 0,$$

where α denotes the minimal acceptable degree, β denotes the maximal degree of rejection and θ denotes the maximal degree of indeterminacy.

The neutrosophic optimization model can be changed into the following optimization model:

$max(\alpha - \beta - \theta)$

Subject to

$$\alpha \leq \mathrm{T}\ (x), \qquad (13)$$
$$\beta \geq \mathrm{F}\ (x),$$
$$\theta \geq \mathrm{I}\ (x),$$
$$\alpha \geq \beta,$$
$$\alpha \geq \theta,$$
$$0 \leq\ \alpha + \beta + \theta \leq 3,$$
$$\alpha, \beta, \theta \geq 0\ ,$$
$$x \geq 0\ .$$

The previous model can be written as:

min (1- α) β θ

Subject to

$$\alpha \leq \mathrm{T}(x)$$
$$\beta \geq \mathrm{F}(x)$$
$$\theta \geq \mathrm{I}(x)$$
$$\alpha \geq \beta$$
$$\alpha \geq \theta$$
$$0 \leq \alpha + \beta + \theta \leq 3 \qquad (14)$$
$$x \geq 0.$$

5 The Algorithm for Solving Neutrosophic Linear Programming Problem (NLPP)

Step 1. solve the objective function subject to the constraints.

Step 2. create the decision set which include the highest degree of truth-membership and the least degree of falsity and indeterminacy memberships.

Step 3. declare goals and tolerance.

Step 4. construct membership functions.

Step 5. set α, β, θ in the interval $]^{-}0, 1^{+}[$ for each neutrosophic number.

Step 6. find the upper and lower bound of objective function as we illustrated previously in section 3.

Step 7. construct neutrosophic optimization model as in equation (13).

6 Numerical Examples

To measure the efficiency of our proposed model, we solved many numerical examples.

6.1. Illustrative Example #1

Beaver Creek Pottery Company is a small crafts operation run by a Native American tribal council. The company employs skilled artisans to produce clay bowls and mugs with authentic Native American designs and colours. The two primary resources used by the company are special pottery clay and skilled labour. Given these limited resources, the company desires to know how many bowls and mugs to produce each day in order to maximize profit. The two products have the following resource requirements for production and profit per item produced presented in Table 1:

Table 1. Resource requirements of two products

product	Resource Requirements		
	Labour(Hr./Unit)	Clay (Lb./Unit)	Profit($/Unit)
Bowl	$\tilde{1}$	$\tilde{4}$	$\widetilde{40}$
Mug	$\tilde{2}$	$\tilde{3}$	$\widetilde{50}$

There are around 40 hours of labour and around 120 pounds of clay available each day for production. We will formulate this problem as a neutrosophic linear programming model as follows:

$$\max \quad \widetilde{40}x_1 + \widetilde{50}x_2$$

S.t.

$$\tilde{1}x_1 + \tilde{2}x_2 \leq \widetilde{40} \qquad \tilde{4}x_1 + \tilde{3}x_2 \leq \widetilde{120}$$

$$x_1, x_2 \geq 0 \tag{15}$$

where

C_1=$\widetilde{40} = \{(30,\ 40,\ 50), (0.7,\ 0.4,\ 0.3)\}$;

C=$\widetilde{50} = \{(40,\ 50,\ 60), (0.6,\ 0.5,\ 0.2)\}$;

a_{11}=$\tilde{1} = \{(0.5,\ 1,\ 3), (0.6,\ 0.4,\ 0.1)\}$;

a_{12}=$\tilde{2} = \{(0,\ 2,\ 6), (0.6,\ 0.4,\ 0.1)\}$;

a_{21}=$\tilde{4} = \{(1,\ 4,\ 12), (0.4,\ 0.3,\ 0.2)\}$;

a_{22}=$\tilde{3} = \{(1, 3,\ 10), (0.7,\ 0.4,\ 0.3)\}$;

b_1=$\widetilde{40} = \{(20,\ 40,\ 60), (0.4,\ 0.3,\ 0.5)\}$;

b_2=$\widetilde{120} = \{(100,\ 120,\ 140), (0.7,\ 0.4,\ 0.3)\}$;

The equivalent crisp formulation is:

$max\ 15x_1 + 18x_2$

S.t

$x_1 + x_2 \leq 12$

$3x_1 + 2x_2 \leq 45$

$x_1, x_2 \geq 0$

The optimal solution is $x_1 = 0$; $x_2 = 12$; with optimal objective value = 216$.

6.2. Illustrative Example #2

$$
\begin{aligned}
&max\ \tilde{5}x_1 + \tilde{3}x_2 \\
&s.t. \\
&\tilde{4}x_1 + \tilde{3}x_2 \le \tilde{12} \\
&\tilde{1}x_1 + \tilde{3}x_2 \le \tilde{6} \\
&x_1, x_2 \ge 0
\end{aligned}
\tag{16}
$$

where

$c_1=\tilde{5}$= {(4, 5, 6), (0.5, 0.8, 0.3)};

$c_2=\tilde{3}$= {(2.5, 3, 3.2), (0.6, 0.4, 0)};

$a_{11}=\tilde{4}$= {(3.5, 4, 4.1), (0.75, 0.5, 0.25)};

$a_{12}=\tilde{3}$= {(2.5, 3, 3.2), (0.2, 0.8, 0.4)};

$a_{21}=\tilde{1}$= {(0, 1, 2), (0.15, 0.5, 0)};

$a_{22}=\tilde{3}$= {(2.8, 3, 3.2), (0.75, 0.5, 0.25)};

$b_1=\tilde{12}$= {(11, 12, 13), (0.2, 0.6, 0.5)};

$b_2=\tilde{6}$= {(5.5, 6, 7.5), (0.8, 0.6, 0.4)}.

The equivalent crisp formulation is:

$max\ 1.3125x_1 + 0.0158x_2$

S.t

$2.5375x_1 + 0.54375x_2 \le 2.475$

$0.3093x_1 + 1.125x_2 \le 2.1375$

$x_1, x_2 \ge 0$

The optimal solution is $x_1 = 1$; $x_2 = 0$; with optimal objective value 1 $.

6.3. Illustrative Example #3

$$
\begin{aligned}
&max\ \tilde{25}x_1 + \tilde{48}x_2 \\
&s.t. \\
&15x_1 + 30x_2 \le 45000 \\
&24x_1 + 6x_2 \le 24000 \\
&21x_1 + 14x_2 \le 28000 \\
&x_1, x_2 \ge 0
\end{aligned} \tag{17}
$$

where

$c_1 = \tilde{25} = \{(19, 25, 33), (0.8, 0.1, 0.4)\}$;

$c_2 = \tilde{48} = \{(44, 48, 54), (0.75, 0.25, 0)\}$.

The corresponding crisp linear programs given as follows:

$$max\ 11.069x_1 + 22.8125x_2$$

$s.t$

$$15x_1 + 30x_2 \le 45000$$

$$24x_1 + 6x_2 \le 24000$$

$x_1, x_2 \ge 0$

The optimal solution is $x_1 = 0$; $x_2 = 1500$; with optimal objective value 34219 $

6.4. Illustrative Example #4

$$
\begin{aligned}
&max\ 25x_1 + 48x_2 \\
&s.t. \\
&\tilde{15}x_1 + \tilde{30}x_2 \le \tilde{45000} \\
&\tilde{24}x_1 + \tilde{6}x_2 \le \tilde{24000} \\
&\tilde{21}x_1 + \tilde{14}x_2 \le \tilde{28000} \\
&x_1, x_2 \ge 0
\end{aligned} \tag{18}
$$

where

$a_{11}= \tilde{15}$= {(14, 15, 17), (0.75, 0.5, 0.25)};

$a_{12}= \tilde{30}$= {(25, 30, 34), (0.25, 0.7, 0.4)};

$a_{21}= \tilde{24}$ = {(21, 24, 26), (0.4, 0.6, 0)};

$a_{22}= \tilde{6}$ = {(4, 6, 8), (0.75, 0.5, 0.25)};

$a_{31}= \tilde{21}$ = {(17, 21, 22), (1, 0.25, 0)};

$a_{32}=\tilde{14}$= {(12, 14, 19), (0.6, 0.4, 0)};

$b_1= \tilde{45000}$= {(44980, 45000, 45030), (0.3, 0.4, 0.8);

$b2= \tilde{24000}$= {(23980, 24000, 24050), (0.4, 0.25, 0.5)};

$b_3= \tilde{28000}$= {(27990, 28000, 28030), (0.9, 0.2, 0)}.

The associated crisp linear programs model will be:

$max\ 25x_1 + 48x_2$

s.t

$5.75x_1 + 6.397x_2 \leq 9282$

$10.312x_1 + 6.187x_2 \leq 14178.37$

$x_1, x_2 \geq 0$

The optimal solution is $x_1 = 0$; $x_2 = 1451$; with optimal objective value 69648$

6.5. Illustrative Example#5

$$\max 7x_1 + 5x_2$$

s.t.

$$\tilde{1}x_1 + \tilde{2}x_2 \leq 6$$

$$\tilde{4}x_1 + \tilde{3}x_2 \leq 12$$

$$x_1, x_2 \geq 0 \qquad (19)$$

where

$a_{11}=\tilde{1}$= {(0.5, 1, 2), (0.2, 0.6, 0.3)};

$a_{12}=\tilde{2}$ = {(2.5, 3, 3.2), (0.6, 0.4, 0.1)};

$a_{21}=\tilde{4}$ = {(3.5, 4, 4.1), (0.5, 0.25, 0.25)};

$a_{22}=\tilde{3}$= {(2.5, 3, 3.2), (0.75, 0.25, 0)};

The associated crisp linear programs model will be:

$max\ 7x_1 + 5x_2$

S. t

$0.284x_1 + 1.142x_2 \leq 6$

$1.45x_1 + 1.36x_2 \leq 12$

$x_1, x_2 \geq 0$

The optimal solution is $x_1 = 4$; $x_2 = 4$; with optimal objective value 48$.

The result of our NLP model in this example is better than the results obtained by intuitionistic fuzzy set [4].

7 Conclusions and Future Work

Neutrosophic sets and fuzzy sets are two hot research topics. In this paper, we propose linear programming model based on neutrosophic environment, simultaneously considering the degrees of acceptance, indeterminacy, and rejection of objectives, by proposed model for solving neutrosophic linear programming problems (NlPP). In the proposed model, we maximize the degrees of acceptance and minimize indeterminacy and rejection of objectives. NlPP was transformed into a crisp programming model using truth membership, indeterminacy membership, and falsity membership functions. We also give numerical examples to show the efficiency of the proposed method. As far as future directions are concerned, these will include studying the duality theory of linear programming problems based on Neutrosophic.

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II

Neutrosophic Linear Fractional Programming Problems

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Abstract

In this chapter, a solution procedure is proposed to solve neutrosophic linear fractional programming (NLFP) problem where cost of the objective function, the resources and the technological coefficients are triangular neutrosophic numbers. Here, the NLFP problem is transformed into an equivalent crisp multi-objective linear fractional programming (MOLFP) problem. By using proposed approach, the transformed MOLFP problem is reduced to a single objective linear programming (LP) problem which can be solved easily by suitable LP problem algorithm.

Keywords

Linear fractional programming; Triangular neutrosophic numbers.

1 Introduction

Linear fractional programming (LFP) is a generalization of linear programming (LP) whereas the objective function in a linear program is a linear function; the objective function in a linear-fractional program is a ratio of two linear functions. Linear fractional programming is used to achieve the highest ratio of profit/cost, inventory/sales, actual cost/standard cost, output/employee, etc. Decision maker may not be able to specify the coefficients (some or all) of LFP problem due to incomplete and imprecise information which tend to be presented in real life situations. Also, aspiration level of objective function and

parameters of problem, hesitate decision maker. These situations can be modeled efficiently through neutrosophic environment. Neutrosophy is the study of neutralities as an extension of dialectics. Neutrosophic is the derivative of neutrosophy and it includes neutrosophic set, neutrosophic probability, neutrosophic statistics and neutrosophic logic. Neutrosophic theory means neutrosophy applied in many fields of sciences, in order to solve problems related to indeterminacy. Although intuitionistic fuzzy sets can only handle incomplete information not indeterminate, the neutrosophic set can handle both incomplete and indeterminate information. [1] Neutrosophic sets characterized by three independent degrees namely truth-membership degree (*T*), indeterminacy-membership degree(*I*), and falsity-membership degree (*F*), where *T,I,F* are standard or non-standard subsets of $]^{-}0, 1^{+}[$. The decision makers in neutrosophic set want to increase the degree of truth-membership and decrease the degree of indeterminacy and falsity membership. The structure of the chapter is as follows: the next section is a preliminary discussion; the third section describes the LFP problem with Charnes and cooper's transformation; the fourth section presents multi-objective linear fractional programming problem; the fifth section presents neutrosophic linear fractional programming problem with solution procedure; the sixth section provides a numerical example to put on view how the approach can be applied; finally, the seventh section provides the conclusion.

2 Preliminaries

In this section, the basic definitions involving neutrosophic set, single valued neutrosophic sets, neutrosophic numbers, triangular neutrosophic numbers and operations on triangular neutrosophic numbers are outlined.

Definition 1. [2]

Let X be a space of points (objects) and $x \in X$. A neutrosophic set A in X is defined by a truth-membership function (x), an indeterminacy-membership function (x) and a falsity-membership function $F(x)$. (x), (x) and $F(x)$ are real standard or real nonstandard subsets of $]^{-}0, 1^{+}[$. That is $TA(x):X \to]^{-}0, 1^{+}[$, $IA(x):X \to]^{-}0, 1^{+}[$ and $FA(x):X \to]^{-}0, 1^{+}[$. There is no restriction on the sum of (x), (x) and $F(x)$, so $0^{-} \leq \sup T_A(x) + \sup I_A(x) + \sup F_A(x) \leq 3^{+}$.

Definition 2. [2]

Let X be a universe of discourse. A single valued neutrosophic set A over X is an object having the form $A=\{\langle x, T_A(x), I_A(x), F_A(x)\rangle : x \in X\}$, where $T_A(x):X \to [0,1]$, $I_A(x):X \to [0,1]$ and $F_A(x):X \to [0,1]$ with $0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$ for all $x \in X$. The intervals $T_A(x)$, $I_A(x)$ and $F_A(x)$ denote the truth-

membership degree, the indeterminacy-membership degree and the falsity membership degree of x to A, respectively. For convenience, a SVN number is denoted by A= (a, b, c), where a, b, $c\in$ [0, 1] and a+b+$c\leq$3.

Definition 3.

Let $\tilde{J}$ be a neutrosophic number in the set of real numbers R, then its truth-membership function is defined as

$$T_{\tilde{J}}(J)=\begin{cases}\dfrac{J-a_1}{a_2-a_1}, & a_1\le J\le a_2,\\ \dfrac{a_2-J}{a_3-a_2}, & a_2\le J\le a_3,\\ 0, & otherwise.\end{cases} \tag{1}$$

Its indeterminacy-membership function is defined as

$$I_{\tilde{J}}(J)=\begin{cases}\dfrac{J-b_1}{b_2-b_1}, & b_1\le J\le b_2,\\ \dfrac{b_2-J}{b_3-b_2}, & b_2\le J\le b_3,\\ 0, & otherwise.\end{cases} \tag{2}$$

And its falsity-membership function is defined as

$$F_{\tilde{J}}(J)=\begin{cases}\dfrac{J-c_1}{c_2-c_1}, & c_1\le J\le c_2,\\ \dfrac{c_2-J}{c_3-c_2}, & c_2\le J\le c_3,\\ 1, & otherwise.\end{cases} \tag{3}$$

Definition 4. [3,9]

A triangular neutrosophic number $a^{\sim n}=<(a_1,b_1,c_1);\propto_a^{\sim n},\theta_a^{\sim n},\beta_a^{\sim n})>$ is a special neutrosophic set on the real number set R, where $\propto_a^{\sim n},\theta_a^{\sim n},\beta_a^{\sim n}\in[0,1]$

The truth-membership, indeterminacy- membership and falsity- membership functions are defined as follows:

$$T_{a^{\sim n}}(x) = \begin{cases} \dfrac{(x-a_1)\,\propto_a^{\sim n}}{(b_1-a_1)} & if\ a_1 \le x \le b_1 \\ \propto_a^{\sim n} & if\ \ x = b_1 \\ \dfrac{(c_1-x)\,\propto_a^{\sim n}}{(c_1-b_1)} & if\ b_1 < x \le c_1 \\ 0 & otherwise \end{cases} \quad (4)$$

$$I_{a^{\sim n}}(x) = \begin{cases} \dfrac{(b_1-x+\theta_a^{\sim n}((x-a_1))}{(b_1-a_1)} & if\ a_1 \le x \le b_1 \\ \theta_a^{\sim n} & if\ \ x = b_1 \\ \dfrac{(x-b_1+\theta_a^{\sim n}\,(c_1-x))}{(c_1-b_1)} & if\ b_1 < x \le c_1 \\ 1 & otherwise \end{cases} \quad (5)$$

$$F_{a^{\sim n}}(x) = \begin{cases} \dfrac{(b_1-x+\beta_a^{\sim n}((x-a_1))}{(b_1-a_1)} & if\ a_1 \le x \le b_1 \\ \beta_a^{\sim n} & if\ \ x = b_1 \\ \dfrac{(x-b_1+\beta_a^{\sim n}\,(c_1-x))}{(c_1-b_1)} & if\ b_1 < x \le c_1 \\ 1 & otherwise \end{cases} \quad (6)$$

If $a_1 \geq 0$ and at least $c_1 > 0$ then:

$$a^{\sim n} =< (a_1, b_1, c_1); \propto_a^{\sim n}, \theta_a^{\sim n}, \beta_a^{\sim n}) >$$

is called a positive triangular neutrosophic number, denoted by $a^{\sim n} > 0$. Likewise, if $c_1 \leq 0$ and at least $a_1 < 0$, then:

$$a^{\sim n} =< (a_1, b_1, c_1); \propto_a^{\sim n}, \theta_a^{\sim n}, \beta_a^{\sim n}) >$$

is called a negative triangular neutrosophic number, denoted by $a^{\sim n} < 0$.

Definition 5. [3,10]

Let

$$a^{\sim n} =< (a_1, b_1, c_1); \propto_a^{\sim n}, \theta_a^{\sim n}, \beta_a^{\sim n}) >,$$

and

$$b^{\sim n} =< (a_2, b_2, c_2); \propto_b^{\sim n}, \theta_b^{\sim n}, \beta_b^{\sim n}) >$$

be two single valued triangular neutrosophic numbers and $\gamma \neq o$ be any real number, then:

$$a^{\sim n} + b^{\sim n} =< (a_1 + a_2\ , b_1 + b_2, c_1 + c_2);\ \propto_a^{\sim n} \wedge \propto_b^{\sim n}, \theta_a^{\sim n} \vee$$

$\theta_b^{\sim n}, \beta_a^{\sim n} \vee \beta_b^{\sim n} >$,

$a^{\sim n}$ - $b^{\sim n} = < (a_1 - c_2, b_1 - b_2, c_1 - a_2); \propto_a^{\sim n} \wedge \propto_b^{\sim n}, \theta_a^{\sim n} \vee \theta_b^{\sim n}, \beta_a^{\sim n} \vee \beta_b^{\sim n} >$,

$$a^{\sim n} b^{\sim n} = \begin{cases} < (a_1 a_2, b_1 b_2, c_1 c_2); \propto_a^{\sim n} \wedge \propto_b^{\sim n}, \theta_a^{\sim n} \vee \theta_b^{\sim n}, \beta_a^{\sim n} \vee \beta_b^{\sim n} > & (c_1 > 0, c_2 > 0) \\ < (a_1 c_2, b_1 b_2, c_1 a_2); \propto_a^{\sim n} \wedge \propto_b^{\sim n}, \theta_a^{\sim n} \vee \theta_b^{\sim n}, \beta_a^{\sim n} \vee \beta_b^{\sim n} > & (c_1 < 0, c_2 > 0) \\ < (c_1 c_2, b_1 b_2, a_1 a_2); \propto_a^{\sim n} \wedge \propto_b^{\sim n}, \theta_a^{\sim n} \vee \theta_b^{\sim n}, \beta_a^{\sim n} \vee \beta_b^{\sim n} > & (c_1 < 0, c_2 < 0) \end{cases}$$

$$\gamma a^{\sim n} = \begin{cases} < (\gamma a_1, \gamma b_1, \gamma c_1); \propto_a^{\sim n}, \theta_a^{\sim n}, \beta_a^{\sim n} > & (\gamma > 0) \\ < (\gamma c_1, \gamma b_1, \gamma a_1); \propto_a^{\sim n}, \theta_a^{\sim n}, \beta_a^{\sim n} > & (\gamma < 0) \end{cases}$$

$$a^{\sim n-1} = < \left(\frac{1}{c_1}, \frac{1}{b_1}, \frac{1}{a_1}\right); \propto_a^{\sim n}, \theta_a^{\sim n}, \beta_a^{\sim n} > (a^{\sim n} \neq 0)$$

$$a^{\sim n} / b^{\sim n} = \begin{cases} < (a_1 / c_2, b_1 / b_2, c_1 / a_2); \propto_a^{\sim n} \wedge \propto_b^{\sim n}, \theta_a^{\sim n} \vee \theta_b^{\sim n}, \beta_a^{\sim n} \vee \beta_b^{\sim n} > & (c_1 > 0, c_2 > 0) \\ < (c_1 / c_2, b_1 / b_2, a_1 / a_2); \propto_a^{\sim n} \wedge \propto_b^{\sim n}, \theta_a^{\sim n} \vee \theta_b^{\sim n}, \beta_a^{\sim n} \vee \beta_b^{\sim n} > & (c_1 < 0, c_2 > 0) \\ < (c_1 / a_2, b_1 / b_2, a_1 / c_2); \propto_a^{\sim n} \wedge \propto_b^{\sim n}, \theta_a^{\sim n} \vee \theta_b^{\sim n}, \beta_a^{\sim n} \vee \beta_b^{\sim n} > & (c_1 < 0, c_2 < 0) \end{cases}$$

3 Linear Fractional Programming Problem (LFPP)

In this section, the general form of LFP problem is discussed. Also, Charnes and Cooper's [4] linear transformation is summarized.

The linear fractional programming (LFP) problem can be written as:

$$\text{Max } Z(x) = \frac{\sum c_j x_j + p}{\sum d_j x_j + q} = \frac{c^T x + p}{d^T x + q} = \frac{N(x)}{D(x)}, \qquad (7)$$

Subject to

$$x \in s = \{x \in R^n : Ax = b, x \geq 0\},$$

where $j = 1,2, \ldots, n, A \in R^{m \times n}$, $b \in R^m$, $c_j, d_j \in R^n$, and $p, q \in R$. For some values of x, $D(x)$ may be equal to zero. To avoid such cases, we require that either $\{Ax \leq b, x \geq 0 \Rightarrow D(x) > 0\}$ or $\{Ax \leq b, x \geq 0 \Rightarrow D(x) < 0\}$. For convenience here, we consider the first case, i.e.

$$\{AX = b, x \geq 0, D(x) > 0\} \qquad (8)$$

Using Charnes and Cooper's linear transformation the previous LFP problem is equivalent to the following linear programming (LP) problem:

Max $c^T y + pt$,

Subject to

$$d^T y + qt = 1, \tag{9}$$
$$Ay - bt = 0,$$
$$t \geq 0, y \geq 0, y \in R^n, t \in R.$$

Consider the fractional programming problem

Max Z(x) = $\frac{N(x)}{D(x)}$, (10)

Subject to

$$Ax \leq b, x \geq 0,$$
$$x \in \Delta = \{x: Ax \leq b, x \geq 0, D(x) > 0$$

$x \in \Delta = \{x: Ax \leq b, x \geq 0 \Rightarrow D(x) > 0\}$

By the transformation $t = \frac{1}{D(x)}$, $y = tx$ we obtained the following:

Max $tN\left(\frac{y}{t}\right)$,

Subject to

$$A\left(\frac{y}{t}\right) - b \leq 0,$$
$$tD\left(\frac{y}{t}\right) = 1,$$
$$t > 0, y \geq 0. \tag{11}$$

By replacing the equality constraint $tD\left(\frac{y}{t}\right) = 1$ by an inequality constraint $tD\left(\frac{y}{t}\right) \leq 1$

We obtained the following:

Max $tN\left(\frac{y}{t}\right)$,

Subject to

$$A\left(\frac{y}{t}\right) - b \leq 0,$$
$$tD\left(\frac{y}{t}\right) \leq 1,$$

$$t > 0, y \geq 0. \tag{12}$$

If in equation 10, $N(x)$ is concave, $D(x)$ is concave and positive on Δ, and $N(x)$ is negative for each $x \in \Delta$, then $Max_{x\in\Delta} \frac{N(x)}{D(x)} \Leftrightarrow Min_{x\in\Delta} \frac{-N(x)}{D(x)} \Leftrightarrow Max_{x\in\Delta} \frac{D(x)}{-N(x)}$, where $-N(x)$ is convex and positive. Now linear fractional program (10) transformed to the following LP problem:

Max $tD\left(\frac{y}{t}\right)$,

Subject to

$$A\left(\frac{y}{t}\right) - b \leq 0,$$

$$-tN\left(\frac{y}{t}\right) \leq 1,$$

$$t > 0, y \geq 0. \tag{13}$$

4 Multi-objective Linear Fractional Programming Problem

In this section, the general form of MOLFP problem is discussed and the procedure for converting MOLFP problem into MOLP problem is illustrated.

The MOLFP problem can be written as follows:

Max $z(x) = [z_1(x), z_2(x), \dots, z_k(x)]$,

Subject to

$$x \in \Delta = \{x : Ax \leq b, x \geq 0\} \tag{14}$$

With $b \in R^m, A \in R^{m\times n}, and\ z_i(x) = \frac{c_i x + p_i}{d_i x + q_i} = \frac{N_i(x)}{D_i(x}, c_i, d_i \in R^n$ and $p_i, q_i \in R, i = 1,2, \dots, k.$

Let *I* be the index set such that $I = \{i:\ N_i(x) \geq 0\ for\ x \in \Delta\}$ and

$I^c = \{i:\ N_i(x) < 0\ for\ x \in \Delta\}, where\ I \cup I^c = \{1,2, \dots, K\}.$ Let $D(x)$ be positive on Δ where Δ is non-empty and bounded. For simplicity, let us take the least value of 1/ $(d_i x + q_i)$ and 1/ [-$(c_i x + p_i)$] is t for $i \in I$ and $i \in I^c$, respectively i.e.

$$\frac{1}{(d_i x + q_i)} \geq t \text{ for } i \in I \text{ and } \frac{-1}{(c_i x + p_i)} \geq t \text{ for } i \in I^c \tag{15}$$

By using the transformation $y = tx\ (t > 0)$, and equation 15, MOLFP problem (14) may be written as follows:

Max $z_i(y,t)$=$\{ tN_i\left(\frac{y}{t}\right)$, for $i \in I$; $tD_i\left(\frac{y}{t}\right)$, for $i \in I^c\}$

Subject to

$$tD_i\left(\frac{y}{t}\right) \leq 1, \text{ for } i \in I,$$

$$- tN_i\left(\frac{y}{t}\right) \leq 1, \text{ for } i \in I^c, \tag{16}$$

$$A\left(\frac{y}{t}\right) - b \leq 0,$$

$$t, y \geq 0\ .$$

If $i \in I$, then truth- membership function of each objective function can be written as:

$$T_i\left(tN_i\left(\frac{y}{t}\right)\right) = \begin{cases} 0 & if\ tN_i\left(\frac{y}{t}\right) \leq 0, \\ \frac{tN_i\left(\frac{y}{t}\right)}{z_i - a_i} & if 0 \leq tN_i\left(\frac{y}{t}\right) \leq z_i + a_i\ , \\ 1 & if\ tN_i\left(\frac{y}{t}\right) \geq z_i + a_i \end{cases} \tag{17}$$

If $i \in I^c$, then truth- membership function of each objective function can be written as:

$$T_i\left(tD_i\left(\frac{y}{t}\right)\right) = \begin{cases} 0 & if\ tD_i\left(\frac{y}{t}\right) \leq 0, \\ \frac{tD_i\left(\frac{y}{t}\right)}{z_i - a_i} & if 0 \leq tD_i\left(\frac{y}{t}\right) \leq z_i + a_i\ , \\ 1 & if\ tD_i\left(\frac{y}{t}\right) \geq z_i + a_i \end{cases} \tag{18}$$

If $i \in I$, then falsity- membership function of each objective function can be written as:

$$F_i\left(tN_i\left(\frac{y}{t}\right)\right) = \begin{cases} 1 & if\ tN_i\left(\frac{y}{t}\right) \leq 0, \\ 1 - \frac{tN_i\left(\frac{y}{t}\right)}{z_i - c_i} & if 0 \leq tN_i\left(\frac{y}{t}\right) \leq z_i + c_i\ , \\ 0 & if\ tN_i\left(\frac{y}{t}\right) \geq z_i + c_i \end{cases} \tag{19}$$

If $i \in I^c$, then falsity- membership function of each objective function can be written as:

$$F_i\left(tD_i\left(\frac{y}{t}\right)\right) = \begin{cases} 1 & if\ tD_i\left(\frac{y}{t}\right) \leq 0, \\ 1 - \frac{tD_i\left(\frac{y}{t}\right)}{z_i - c_i} & if 0 \leq tD_i\left(\frac{y}{t}\right) \leq z_i + c_i\ , \\ 0 & if\ tD_i\left(\frac{y}{t}\right) \geq z_i + c_i \end{cases} \tag{20}$$

If $i \in I$, then indeterminacy- membership function of each objective function can be written as

$$I_i\left(tN_i\left(\frac{y}{t}\right)\right) = \begin{cases} 0 & if\ tN_i\left(\frac{y}{t}\right) \leq 0, \\ \frac{tN_i\left(\frac{y}{t}\right)}{z_i - d_i} & if 0 \leq tN_i\left(\frac{y}{t}\right) \leq z_i + d_i\ , \\ 0 & if\ tN_i\left(\frac{y}{t}\right) \geq z_i + d_i \end{cases} \tag{21}$$

If $i \in I^c$, then indeterminacy - membership function of each objective function can be written as:

$$I_i\left(tD_i\left(\frac{y}{t}\right)\right) = \begin{cases} 0 & if\ tD_i\left(\frac{y}{t}\right) \leq 0, \\ \frac{tD_i\left(\frac{y}{t}\right)}{z_i - d_i} & if 0 \leq tD_i\left(\frac{y}{t}\right) \leq z_i + d_i\ , \\ 0 & if\ tD_i\left(\frac{y}{t}\right) \geq z_i + d_i \end{cases} \tag{22}$$

where, a_i , d_i and c_i are acceptance tolerance, indeterminacy tolerance and rejection tolerance.

Zimmermann [5] proved that if membership function $\mu_D(y,t)$ of complete solution set(y,t), has a unique maximum value$\mu_D(y^*,t^*)$ then (y^*,t^*) which is an element of complete solution set (y,t) can be derived by solving linear programming with one variableλ.

Using Zimmermann's min operator and membership functions, the model (14) transformed to the crisp model as:

Maxλ,

Subject to

$$T_i\left(tN_i\left(\frac{y}{t}\right)\right) \geq \lambda \text{ , for } i \in I$$

$$T_i\left(tD_i\left(\frac{y}{t}\right)\right) \geq \lambda \text{ , for } i \in I^c$$

$$F_i\left(tN_i\left(\frac{y}{t}\right)\right) \leq \lambda \text{ , for } i \in I$$

$$F_i\left(tD_i\left(\frac{y}{t}\right)\right) \leq \lambda \text{ , for } i \in I^c$$

$$I_i\left(tN_i\left(\frac{y}{t}\right)\right) \leq \lambda \text{ , for } i \in I$$

$$I_i\left(tD_i\left(\frac{y}{t}\right)\right) \leq \lambda \text{ , for } i \in I^c$$

$$tD_i\left(\frac{y}{t}\right) \leq 1, \quad \text{for } i \in I,$$

$$-tN_i\left(\frac{y}{t}\right) \leq 1, \quad \text{for } i \in I^c, \tag{23}$$

$$A\left(\frac{y}{t}\right) - b \leq 0,$$

$$t, y, \lambda \geq 0 .$$

5 Neutrosophic Linear Fractional Programming Problem

In this section, we propose a procedure for solving neutrosophic linear fractional programming problem where the cost of the objective function, the resources, and the technological coefficients are triangular neutrosophic numbers.

Let us consider the NLFP problem:

$$\text{Max } Z(x^{\sim n}) = \frac{\sum c_j^{\sim n} x_j + p^{\sim n}}{\sum d_j^{\sim n} x_j + q^{\sim n}}$$

Subject to

$$\sum a_{ij}^{\sim n} x_j \leq b_i^{\sim n}, i = 1,2,\dots,m, \tag{24}$$

$$x_j \geq 0, j = 1,2,\dots,n.$$

We assume that $c_j^{\sim n}$, $p^{\sim n}, d_j^{\sim n}, q^{\sim n}, a_{ij}^{\sim n}$ *and* $b_i^{\sim n}$ are triangular neutrosophic numbers for each $i = 1,2,\dots,m$ and $j = 1,2,\dots,n.$ therefore, the

problem (24) can be written as: (25)

$$\text{Max } Z(x^{\sim n}) = \frac{\sum(c_{j1}, c_{j2}, c_{j3};\ \alpha_c, \theta_c, \beta_c)x_j + (p_1, p_2, p_3;\ \alpha_p, \theta_p, \beta_p)}{\sum(d_{j1}, d_{j2}, d_{j3};\ \alpha_d, \theta_d, \beta_d)x_j + (q_1, q_2, q_3;\ \alpha_q, \theta_q, \beta_q)}$$

Subject to

$$\sum(a_{ij1}, a_{ij2}, a_{ij3};\ \alpha_a, \theta_a, \beta_a)x_j \leq (b_{i1}, b_{i2}, b_{i3};\ \alpha_b, \theta_b, \beta_b), i = 1,2,\dots,m,$$

$$x_j \geq 0, j = 1,2,\dots,n.$$

where $\propto, \theta, \beta \in [0,1]$ and stand for truth-membership, indeterminacy and falsity-membership function of each neutrosophic number.

Here decision maker wants to increase the degree of truth-membership and decrease the degree of indeterminacy and falsity membership. Using the concept of component wise optimization, the problem (25) reduces to an equivalent MOLFP as follows:

$$\text{Max } Z_1(x) = \frac{\sum c_{j1}\ x_j + p_1}{\sum d_{j3}\ x_j + q_3},$$

$$\text{Max } Z_2(x) = \frac{\sum c_{j2}\ x_j + p_2}{\sum d_{j2}\ x_j + q_2},$$

$$\text{Max } Z_3(x) = \frac{\sum c_{j3}\ x_j + p_3}{\sum d_{j1}\ x_j + q_1}, \quad (26)$$

$$\text{Max } Z_4(x) = \frac{\sum \alpha_c x_j + \alpha_p}{\sum \beta_d x_j + \beta_q},$$

$$\text{Max } Z_5(x) = 1 - \frac{\sum \theta_c x_j + \theta_p}{\sum \theta_d x_j + \theta_q},$$

$$\text{Max } z_6(x) = 1 - \frac{\sum \beta_c x_j + \beta_p}{\sum \alpha_d x_j + \alpha_q},$$

Subject to

$$\sum a_{ij1} x_j \leq b_{i1},$$

$$\sum a_{ij2} x_j \leq b_{i2},$$

$$\sum a_{ij3} x_j \leq b_{i3},$$

$$\sum \alpha_a x_j \leq \alpha_b ,$$

$$\sum \theta_{a,} x_j \leq \theta_b ,$$

$$\sum \beta_{a,} x_j \leq \beta_b ,$$

$$x_j \geq 0, i = 1,2, \dots, m;\ j = 1,2, \dots, n.$$

Let us assume that z_1, z_2, z_3, z_4, z_5 and $z_6 \geq 0$ for the feasible region. Hence, the MOLFP problem can be converted into the following MOLP problem:

$$\text{Max } z_1(y,t) = \sum c_{j1}\ y_j + p_1 t,$$

$$\text{Max } z_2(y,t) = \sum c_{j2}\ y_j + p_2 t,$$

$$\text{Max } z_3(y,t) = \sum c_{j3}\ y_j + p_3 t,$$

$$\text{Max } z_4(y,t) = \sum \alpha_c y_j + \alpha_p\ t,$$

$$\text{Max } z_5(y,t) = 1 - \left(\sum \theta_c y_j + \theta_p t\right),$$

$$\text{Max } z_6(y,t) = 1 - \left(\sum \beta_c y_j + \beta_p t\right),$$

Subject to

$$\sum d_{j3}\ y_j + q_3 t \leq 1,$$

$$\sum d_{j2}\ y_j + q_2 t \leq 1,$$

$$\sum d_{j1}\ y_j + q_1 t \leq 1,$$

$$\sum \beta_d y_j + \beta_q t \leq 1,$$

$$\sum \theta_{d,} y_j + \theta_q t \leq 1,$$

$$\sum \alpha_d y_j + \alpha_q t \leq 1,$$

$$\sum a_{ij1}\ y_j - b_{i1} t \leq 0,$$

$$\sum a_{ij2}\ y_j - b_{i2} t \leq 0,$$

$$\sum a_{ij3}\, y_j \; - b_{i3} t \leq 0,$$

$$\sum \alpha_a \; y_j \; - \alpha_b \; t \leq 0,$$

$$\sum \theta_a \; y_j \; - \theta_{b,} t \leq 0, \qquad (27)$$

$$\sum \beta_a \; y_j \; - \beta_b \; t \leq 0,$$

$$t, y_j \; \geq 0, i = 1,2, \dots, m; j = 1,2, \dots, n.$$

Solving the transformed MOLP problem for each objective function, we obtain $z_1^*, z_2^*, z_3^*, z_{4,}^* z_5^*$ and z_6^*.

Using the membership functions defined in previous section, the above model reduces to:

Max λ,

Subject to

$$\sum c_{j1} \; y_j \; + p_1 \, t - z_1^* \lambda \geq 0,$$

$$\sum c_{j2} \; y_j \; + p_2 \, t - z_2^* \lambda \geq 0,$$

$$\sum c_{j3} \; y_j \; + p_3 \, t - z_3^* \lambda \geq 0,$$

$$\sum \alpha_c \, y_j \; + \alpha_p \, t - z_4^* \lambda \geq 0,$$

$$1 - \left(\sum \theta_c \, y_j \; + \theta_p \, t \right) - z_5^* \lambda \leq 0,$$

$$1 - \left(\sum \beta_c y_j \; + \beta_p \, t \right) - z_6^* \lambda \leq 0,$$

$$\sum d_{j3} \; y_j \; + q_3 \, t \leq 1,$$

$$\sum d_{j2} \; y_j \; + q_2 \, t \leq 1,$$

$$\sum d_{j1} \; y_j \; + q_1 \, t \leq 1,$$

$$\sum \beta_d y_j + \beta_q t \leq 1,$$

$$\sum \theta_{d,} y_j + \theta_q t \leq 1,$$

$$\sum \alpha_d y_j + \alpha_q t \leq 1,$$

$$\sum a_{ij1}\, y_j - b_{i1} t \leq 0,$$

$$\sum a_{ij2}\, y_j - b_{i2} t \leq 0,$$

$$\sum a_{ij3}\, y_j - b_{i3} t \leq 0,$$

$$\sum \alpha_a\, y_j - \alpha_b\, t \leq 0,$$

$$\sum \theta_a\, y_j - \theta_{b,} t \leq 0,$$

$$\sum \beta_a\, y_j - \beta_{b,} t \leq 0, t, y_j, \lambda \geq 0, i = 1,2, \dots, m; j = 1,2, \dots, n$$

$$t, y_j, \lambda \geq 0, i = 1,2, \dots, m; j = 1,2, \dots, n. \quad (28)$$

5.1 Algorithm

The proposed approach for solving NLFP problem can be summarized as follows:

Step 1. The NLFP problem is converted into MOLFP problem using componentwise optimization of triangular neutrosophic numbers.

Step 2. The MOLFP problem is transformed into MOLP problem using the method proposed by Charnes and Cooper.

Step 3. Solve each objective function subject to the given set of constraints.

Step 4. Define membership functions for each objective function as in section four.

Step 5. Use Zimmermann's operator and membership functions to obtain crisp model.

Step 6. Solve crisp model by using suitable algorithm.

6 Numerical Example

A company manufactures 3 kinds of products I, II and III with profit around 8, 7 and 9 dollars per unit, respectively. However, the cost for each one unit of

the products is around 8, 9 and 6 dollars, respectively. Also it is assumed that a fixed cost of around 1.5 dollars is added to the cost function due to expected duration through the process of production. Suppose the materials needed for manufacturing the products I, II and III are about 4, 3 and 5 units per pound, respectively. The supply for this raw material is restricted to about 28 pounds. Man-hours availability for product I is about 5 hours, for product II is about 3 hours, and that for III is about 3 hours in manufacturing per units. Total man-hours availability is around 20 hours daily. Determine how many products of I, II and III should be manufactured in order to maximize the total profit. Also during the whole process, the manager hesitates in prediction of parametric values due to some uncontrollable factors.

Let x_1, x_2 and x_3 units be the amount of I, II and III, respectively to be produced. After prediction of estimated parameters, the above problem can be formulated as the following NLFPP:

$$\text{Max } Z(x^{\sim n}) = \frac{8^{\sim n}x_1 + 7^{\sim n}x_2 + 9^{\sim n}x_3}{8^{\sim n}x_1 + 9^{\sim n}x_2 + 6^{\sim n}x_3 + 1.5^{\sim n}}$$

Subject to

$$\begin{aligned} &4^{\sim n}x_1 + 3^{\sim n}x_2 + 5^{\sim n}x_3 \leq 28^{\sim n}, \\ &5^{\sim n}x_1 + 3^{\sim n}x_2 + 3^{\sim n}x_3 \leq 20^{\sim n}, \\ &x_1, x_2, x_3 \geq 0. \end{aligned} \tag{29}$$

with

$$8^{\sim n} = (7,8,9; 0.5,0.8,0.3),\ 7^{\sim n} = (6,7,8; 0.2,0.6,0.5),$$
$$9^{\sim n} = (8,9,10;\ 0.8,0.1,0.4),$$
$$6^{\sim n} = (4,6,8; 0.75,0.25,0.1),\ 1.5^{\sim n} = (1,1.5,2;\ 0.75,0.5,0.25),$$
$$4^{\sim n} = (3,4,5; 0.4,0.6,0.5),\ 3^{\sim n} = (2,3,4;\ 1,0.25,0.3),$$
$$5^{\sim n} = (4,5,6; 0.3,0.4,0.8),\qquad 28^{\sim n} = (25,28,30;\ 0.4,0.25,0.6),$$
$$20^{\sim n} = (18,20,22; 0.9,0.2,0.6).$$

This problem is equivalent to the following MOLFPP:

$$\text{Max} z_1(x) = \frac{7x_1 + 6x_2 + 8x_3}{9x_1 + 10x_2 + 8x_3 + 2},$$

$$\text{Max} z_2(x) = \frac{8x_1 + 7x_2 + 9x_3}{8x_1 + 9x_2 + 6x_3 + 1.5},$$

$$\text{Max} z_3(x) = \frac{9x_1 + 8x_2 + 10x_3}{7x_1 + 8x_2 + 4x_3 + 1},$$

$$\text{Max} z_4(x) = \frac{0.5x_1+0.2x_2+0.8x_3}{0.3x_1+0.4x_2+0.1x_3+0.25},$$

$$\text{Max} z_5(x) = 1 - \frac{0.8x_1+0.6x_2+0.1x_3}{0.8x_1+0.1x_2+0.25x_3+0.5},$$

$$\text{Max} z_6(x) = 1 - \frac{0.3x_1+0.5x_2+0.4x_3}{0.5x_1+0.8x_2+0.75x_3+0.75}, \quad (30)$$

Subject to

$$3x_1 + 2x_2 + 4x_3 \leq 25,$$

$$4x_1 + 3x_2 + 5x_3 \leq 28,$$

$$5x_1 + 4x_2 + 6x_3 \leq 30,$$

$$4x_1 + 2x_2 + 2x_3 \leq 18,$$

$$5x_1 + 3x_2 + 3x_3 \leq 20,$$

$$6x_1 + 4x_2 + 4x_3 \leq 22,$$

$$0.4x_1 + x_2 + 0.3x_3 \leq 0.4,$$

$$0.6x_1 + 0.25x_2 + 0.4x_3 \leq 0.25,$$

$$0.5x_1 + 0.3x_2 + 0.8x_3 \leq 0.5,$$

$$0.3x_1 + x_2 + x_3 \leq 0.9,$$

$$0.4x_1 + 0.25x_2 + 0.25x_3 \leq 0.2,$$

$$0.8x_1 + 0.3x_2 + 0.3x_3 \leq 0.6,$$

Using the transformation, the problem is equivalent to the following MOLPP:

$$\text{Max } z_1(y,t) = 7y_1 + 6y_2 + 8y_3,$$

$$\text{Max } z_2(y,t) = 8y_1 + 7y_2 + 9y_3,$$

$$\text{Max } z_3(y,t) = 9y_1 + 8y_2 + 10y_3,$$

$$\text{Max } z_4(y,t) = 0.5y_1 + 0.2y_2 + 0.8y_3,$$

$$\text{Max } z_5(y,t) = 0.5y_2 + 0.15y_3 + 0.5,$$

$$\text{Max } z_6(y,t) = 0.2y_1 + 0.3y_2 + 0.35y_3 + 0.75, \quad (31)$$

Subject to

$$9y_1 + 10y_2 + 8y_3 + 2t \leq 1,$$

$$8y_1 + 9y_2 + 6y_3 + 1.5t \leq 1,$$

$$7y_1 + 8y_2 + 4y_3 + t \leq 1,$$

$0.3y_1 + 0.4y_2 + 0.1y_3 + 0.25t \leq 1$,

$0.8y_1 + 0.1y_2 + 0.25y_3 + 0.5t \leq 1$,

$0.5y_1 + 0.8y_2 + 0.75y_3 + 0.75t \leq 1$,

$3y_1 + 2y_2 + 4y_3 - 25t \leq 0$,

$4y_1 + 3y_2 + 5y_3 - 28t \leq 0$,

$5y_1 + 4y_2 + 6y_3 - 30t \leq 0$,

$4y_1 + 2y_2 + 2y_3 - 18t \leq 0$,

$5y_1 + 3y_2 + 3y_3 - 20t \leq 0$,

$6y_1 + 4y_2 + 4y_3 - 22t \leq 0$,

$0.4y_1 + y_2 + 0.3y_3 - 0.4t \leq 0$,

$0.6y_1 + 0.25y_2 + 0.4y_3 - 0.25t \leq 0$,

$0.5y_1 + 0.3y_2 + 0.8y_3 - 0.5t \leq 0$,

$0.3y_1 + y_2 + y_3 - 0.9t \leq 0$,

$y_1, y_2, y_3 \geq 0, \quad t > 0.$

Solving each objective at a time we get

Z1= 0.7143

Z2=0.8036

Z3=0.8929

Z4=0.0714

Z5=0.833

Z6=0.7813.

$z_1 = \quad , z_2 = \quad , z_3 = \quad , z_4 = \quad , z_5 = \quad , z_6 =$

Now the previous problem can be reduced to the following LPP:

Max λ

Subject to

$7y_1 + 6y_2 + 8y_3 - z_1 \lambda \geq 0,$

$8y_1 + 7y_2 + 9y_3 - z_2 \lambda \geq 0,$

$9y_1 + 8y_2 + 10y_3 - z_3 \lambda \geq 0,$

$$
\begin{aligned}
&0.5y_1 + 0.2y_2 + 0.8y_3 - z_4\ \lambda \geq 0,\\
&0.5y_2 + 0.15y_3 + 0.5 - z_5\ \lambda \leq 0,\\
&0.2y_1 + 0.3y_2 + 0.3y_3 + 0.75 - z_6\ \lambda \leq 0,\\
&9y_1 + 10y_2 + 8y_3 + 2t \leq 1,\\
&8y_1 + 9y_2 + 6y_3 + 1.5t \leq 1,\\
&7y_1 + 8y_2 + 4y_3 + t \leq 1,\\
&0.3y_1 + 0.4y_2 + 0.1y_3 + 0.25t \leq 1,\\
&0.8y_1 + 0.1y_2 + 0.25y_3 + 0.5t \leq 1,\\
&0.5y_1 + 0.8y_2 + 0.75y_3 + 0.75t \leq 1,\\
&3y_1 + 2y_2 + 4y_3 - 25t \leq 0,\\
&4y_1 + 3y_2 + 5y_3 - 28t \leq 0,\\
&5y_1 + 4y_2 + 6y_3 - 30t \leq 0,\\
&4y_1 + 2y_2 + 2y_3 - 18t \leq 0,\\
&5y_1 + 3y_2 + 3y_3 - 20t \leq 0,\\
&6y_1 + 4y_2 + 4y_3 - 22t \leq 0,\\
&0.4y_1 + y_2 + 0.3y_3 - 0.4t \leq 0,\\
&0.6y_1 + 0.25y_2 + 0.4y_3 - 0.25t \leq 0,\\
&0.5y_1 + 0.3y_2 + 0.8y_3 - 0.5t \leq 0,\\
&0.3y_1 + y_2 + y_3 - 0.9t \leq 0,\\
&y_1, y_2, y_3 \geq 0, \qquad t > 0.
\end{aligned}
\tag{32}
$$

Solving by LINGO we have

$y_1=0$

$y_2=0$

$y_3=0.0893$

$t=0.1429$

$\lambda=1$

7 Conclusion

In this chapter, a method for solving the NLFP problems where the cost of the objective function, the resources and the technological coefficients are triangular neutrosophic numbers, is proposed. In the method, NLFP problem is transformed to a MOLFP problem and the resultant problem is converted to a LP problem. In future, the proposed approach can be extended for solving multi-objective neutrosophic linear fractional programming problems (MONLFPPs).

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III

Neutrosophic Integer Programming Problems

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Abstract

In this chapter, we introduce the integer programming in neutrosophic environment, by considering coefficients of problem as a triangular neutrosophic numbers. The degrees of acceptance, indeterminacy and rejection of objectives are simultaneously considered. The Neutrosophic Integer Programming Problem (NIP) is transformed into a crisp programming model, using truth membership (T), indeterminacy membership (I), and falsity membership (F) functions as well as single valued triangular neutrosophic numbers. To measure the efficiency of our proposed model we solved several numerical examples**.**

Keywords

Neutrosophic; integer programming; single valued triangular neutrosophic number.

1 Introduction

In linear programming models, decision variables are allowed to be fractional. For example, it is reasonable to accept a solution giving an hourly production of automobiles at 64.5, if the model were based upon average hourly production. However, fractional solutions are not realistic in many situations and to deal with this matter, integer programming problems are introduced. We can define integer programming problem as a linear programming problem with

integer restrictions on decision variables. When some, but not all decision variables are restricted to be integer, this problem called a mixed integer problem and when all decision variables are integers, it's a pure integer program. Integer programming plays an important role in supporting managerial decisions. In integer programming problems, the decision maker may not be able to specify the objective function and/or constraints functions precisely. In 1995, Smarandache [1-3] introduce neutrosophy which is the study of neutralities as an extension of dialectics. Neutrosophic is the derivative of neutrosophy and it includes neutrosophic set, neutrosophic probability, neutrosophic statistics and neutrosophic logic. Neutrosophic theory means neutrosophy applied in many fields of sciences, in order to solve problems related to indeterminacy. Although intuitionistic fuzzy sets can only handle incomplete information not indeterminate, the neutrosophic set can handle both incomplete and indeterminate information. [4] Neutrosophic sets characterized by three independent degrees as in Fig. 1., namely truth-membership degree (T), indeterminacy-membership degree(I), and falsity-membership degree (F), where T,I,F are standard or non-standard subsets of $]^{-}0,\ 1^{+}[$. The decision makers in neutrosophic set want to increase the degree of truth-membership and decrease the degree of indeterminacy and falsity membership.

The structure of the chapter is as follows: the next section is a preliminary discussion; the third section describes the formulation of integer programing problem using the proposed model; the fourth section presents some illustrative examples to put on view how the approach can be applied; the last section summarizes the conclusions and gives an outlook for future research.

2 Preliminaries

2.1 Neutrosophic Set [4]

Let X be a space of points (objects) and $x\in X$. A neutrosophic set A in X is defined by a truth-membership function $T(x)$, an indeterminacy-membership function $I(x)$ and a falsity-membership function (x). $T(x)$, $I(x)$ and (x) are real standard or real nonstandard subsets of $]0^{-},1^{+}[$. That is $T_A(x):X\rightarrow]0^{-},1^{+}[$, $I_A(x):X\rightarrow]0^{-},1^{+}[$ and $F_A(x):X\rightarrow]0^{-},1^{+}[$. There is no restriction on the sum of $T(x)$, $I(x)$ and $F(x)$, so $0^{-}\leq\sup(x)\leq\sup I_A(x)\leq F_A(x)\leq 3^{+}$.

2.2 Single Valued Neutrosophic Sets (SVNS) [3-4]

Let X be a universe of discourse. A single valued neutrosophic set A over X is an object having the form $A=\{\langle x,\ T(x),\ I_A(x),F_A(x)\rangle : x\in X\}$, where $T_A(x):X\rightarrow[0,1]$, $I_A(x):X\rightarrow[0,1]$ and $F_A(x):X\rightarrow[0,1]$ with $0\leq T_A(x)+ I_A(x)+F_A(x)\leq 3$ for all $x\in X$. The intervals $T(x)$, $I(x)$ and $F_A(x)$ denote the truth-membership

degree, the indeterminacy-membership degree and the falsity membership degree of x to A, respectively.

In the following, we write SVN numbers instead of single valued neutrosophic numbers. For convenience, a SVN number is denoted by A= (a, b, c), where a, b, $c\in$ [0, 1] and $a+b+c\leq 3$.

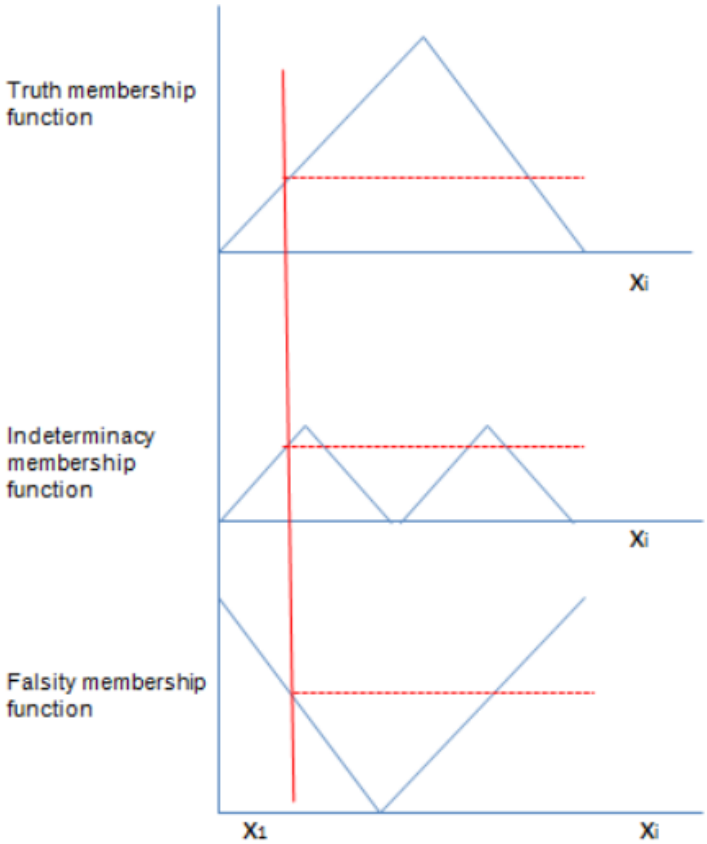

Fig.1: Neutrosophication process

2.3 Complement [5]

The complement of a single valued neutrosophic set A is denoted by $_C$ (A) and is defined by

$$T_c(A)(x) = F(A)(x),$$
$$I_c(A)(x) = 1 - I(A)(x),$$
$$F_c(A)(x) = T(A)(x) \qquad \text{for all } x \text{ in } X$$

2.4 Union [5]

The union of two single valued neutrosophic sets A and B is a single valued neutrosophic set C, written as C = A∪B, whose truth-membership, indeterminacy membership and falsity-membership functions are given by

$$T(C)(x) = max\ (\ T(A)(x)\ ,T(B)(x)\ )\ ,$$

$$I(C)(x) = max\ (\ I(A)(x)\ ,I(B)(x)\ )\ ,$$

$$F(C)(x) =\ min((A)(x)\ ,F(B)(x)\ )\ \text{ for all } x \text{ in } X$$

2.5 Intersection [5]

The intersection of two single valued neutrosophic sets A and B is a single valued neutrosophic set C, written as

C = A∩B, whose truth-membership, indeterminacy membership and falsity-membership functions are given by

$$T(C)(x) = min\ (\ T(A)(x)\ ,T(B)(x)\ )\ ,$$

$$I(C)(x) = min\ (\ I(A)(x)\ ,I(B)(x)\ )\ ,$$

$$F(C)(x) =\ max((A)(x)\ ,F(B)(x)\ )\ \text{ for all } x \text{ in } X$$

3 Neutrosophic Integer Programming Problems

Integer programming problem with neutrosophic coefficients (NIPP) is defined as the following:

Maximize Z= $\sum_{j=1}^{n} \tilde{c}_j x_j$

Subject to

$$\sum_{j=1}^{n} a_{ij}^{\sim n} x_j \le b_i \quad i = 1, \dots, m\ , \tag{1}$$

$$x_j \ge 0, \qquad j = 1, \dots n\ ,$$

x_j Integer for $j \in \{0,1, \dots n\}$.

where $\tilde{c}_j$, $a_{ij}^{\sim n}$ are neutrosophic numbers.

The single valued neutrosophic number ($a_{ij}^{\sim n}$) is donated by A=(a,b,c) where a,b,c ∈ [0,1] And a,b,c ≤ 3

The truth- membership function of neutrosophic number $a_{ij}^{\sim n}$ is defined as:

$$T\ a_{ij}^{\sim n}(x) = \begin{cases} \frac{x-a_1}{a_2-a_1} & a_1 \le x \le a_2 \\ \frac{a_2-x}{a_3-a_2} & a_2 \le x \le a_3 \\ 0 & otherwise \end{cases} \tag{2}$$

The indeterminacy- membership function of neutrosophic number $a_{ij}^{n}$is defined as:

$$\text{I } a_{ij}^{\sim n}(x) = \begin{cases} \frac{x-b_1}{b_2-b_1} & b_1 \le x \le b_2 \\ \frac{b_2-x}{b_{3-b_2}} & b_2 \le x \le b_3 \\ 0 \ otherwise \end{cases} \quad (3)$$

And its falsity- membership function of neutrosophic number $a_{ij}^{\sim n}$ is defined as:

$$\text{F } a_{ij}^{\sim n}(x) = \begin{cases} \frac{x-C_1}{C_2-C_1} & C_1 \le x \le C_2 \\ \frac{c_2-x}{c_{3-c_2}} & C_2 \le x \le C_3 \\ 1 \ otherwise \end{cases} \quad (4)$$

Then we find the maximum and minimum values of the objective function for truth-membership, indeterminacy and falsity membership as follows:

$f^{max} = \max\{f(x_i^*)\}$ and $f^{min} = \min\{f(x_i^*)\}$,

where $1 \le i \le k$

$f_{min}^{F} = f_{min}^{T}$ and $f_{max}^{F} = f_{max}^{T} - R\left(f_{max}^{T} - f_{min}^{T}\right)$

$f_{max}^{I} = f_{max}^{I}$ and $f_{min}^{I} = f_{min}^{I} - S\left(f_{max}^{T} - f_{min}^{T}\right)$,

where R, S are predetermined real number in (0, 1)

The truth membership, indeterminacy membership, falsity membership of objective function are as follows:

$$T^f(x) = \begin{cases} 1 & if \ f \le f^{min} \\ \frac{f^{max}-f(x)}{f^{max}-f^{min}} & if \ f^{min} < f(x) \le f^{max} \\ 0 & if \ f(x) > f^{max} \end{cases} \quad (5)$$

$$I^f(x) = \begin{cases} 0 & if \ f \le f^{min} \\ \frac{f(x)-f^{max}}{f^{max}-f^{min}} & if \ f^{min} < f(x) \le f^{max} \\ 0 & if \ f(x) > f^{max} \end{cases} \quad (6)$$

$$F^f(x) = \begin{cases} 0 & if \ f \le f^{min} \\ \frac{f(x)-f^{min}}{f^{max}-f^{min}} & if \ f^{min} < f(x) \le f^{max} \\ 1 & if \ f(x) > f^{max} \end{cases} \quad (7)$$

The neutrosophic set of the j^{th} decision variable x_j is defined as:

$$T_{x_j}^{(x)} = \begin{cases} 1 & if \quad x_j \le 0 \\ \frac{d_j - x_j}{d_j} & if \quad 0 < x_j \le d_j \\ 0 & if \quad x_j > d_j \end{cases} \tag{8}$$

$$F_{x_j}^{(x)} = \begin{cases} 0 & if \quad x_j \le 0 \\ \frac{x_j}{d_j + b_j} & if \quad 0 < x_j \le d_j \\ 1 & if \quad x_j > d_j \end{cases} \tag{9}$$

$$I_j^{(x)} = \begin{cases} 0 & if \quad x_j \le 0 \\ \frac{x_j - d_j}{d_j + b_j} & if \quad 0 < x_j \le d_j \\ 0 & if \quad x_j > d_j \end{cases} \tag{10}$$

where d_j, b_j are integer numbers.

4 Neutrosophic Optimization Model of Integer Programming Problem

In our neutrosophic model we want to maximize the degree of acceptance and minimize the degree of rejection and indeterminacy of the neutrosophic objective function and constraints. Neutrosophic optimization model can be defined as:

$$maxT_{(x)}$$

$$minF_{(x)}$$

$$minI_{(x)}$$

Subject to

$$T_{(X)} \ge F_{(x)}$$

$$T_{(X)} \ge I_{(x)}$$

$$0 \le T_{(X)} + I_{(x)} + F_{(x)} \le 3 \tag{11}$$

$$T_{(X)}, \quad I_{(X)}, \quad F_{(X)} \ge 0$$

$$x \ge 0 \text{ , integer,}$$

where $T_{(x)}. F_{(x)}, I_{(x)}$denotes the degree of acceptance, rejection and indeterminacy of x respectively.

The above problem is equivalent to the following:

$$max\ \alpha,\ min\ \beta\ , min\ \theta$$

Subject to

$\alpha \leq T_{(x)}$

$\beta \leq F_{(x)}$

$\theta \leq I_{(x)}$

$\alpha \geq \beta$

$\alpha \geq \theta$

$0 \leq \alpha + \beta + \theta \leq 3$ (12)

$x \geq 0$, integer,

where α denotes the minimal acceptable degree, β denote the maximal degree of rejection and θ denote maximal degree of indeterminacy.

The neutrosophic optimization model can be changed into the following optimization model:

$max(\alpha - \beta - \theta)$

Subject to

$\alpha \leq T_{(x)}$ (13)

$\beta \geq F_{(x)}$

$\theta \geq I_{(x)}$

$\alpha \geq \beta$

$\alpha \geq \theta$s

$0 \leq \alpha + \beta + \theta \leq 3$

$\alpha, \beta, \theta \geq 0$

$x \geq 0$, integer.

The previous model can be written as:

min (1- α) β θ

Subject to

$\alpha \leq T_{(x)}$

$\beta \geq F_{(x)}$

$\theta \geq I_{(x)}$

$$\alpha \geq \beta$$

$$\alpha \geq \theta$$

$$0 \leq \alpha + \beta + \theta \leq 3 \tag{14}$$

$x \geq 0$, integer.

5 The Algorithms for Solving Neutrosophic Integer Programming Problem (NIPP)

5.1 Neutrosophic Cutting Plane Algorithm

Step 1: Convert neutrosophic integer programming problem to its crisp model by using the following method:

By defining a method to compare any two single valued triangular neutrosophic numbers which is based on the score function and the accuracy function. Let $\tilde{a} = \langle (a_1, b_1, c_1), w_{\tilde{a}}, u_{\tilde{a}}, y_{\tilde{a}} \rangle$ be a single valued triangular neutrosophic number, then

$$S(\tilde{a}) = \frac{1}{16}[a + b + c] \times (2 + \mu_{\tilde{a}} - v_{\tilde{a}} - \lambda_{\tilde{a}}) \tag{15}$$

and

$$A(\tilde{a}) = \frac{1}{16}[a + b + c] \times (2 + \mu_{\tilde{a}} - v_{\tilde{a}} + \lambda_{\tilde{a}}) \tag{16}$$

It is called the score and accuracy degrees of $\tilde{a}$, respectively. The neutrosophic integer programming NIP can be represented by crisp programming model using truth membership, indeterminacy membership, and falsity membership functions and the score and accuracy degrees of ã, at equations (15) or (16).

Step 2: Create the decision set which include the highest degree of truth-membership and the least degree of falsity and indeterminacy memberships.

Step 3: Solve the problem as a linear programming problem and ignore integrality.

Step 4: If the optimal solution is integer, then it's right. Otherwise, go to the next step.

Step 5: Generate a constraint which is satisfied by all integer solutions and add this constraint to the problem.

Step 6: Go to step 1.

5.2 Neutrosophic Branch and Bound Algorithm

Step 1: Convert neutrosophic integer programming problem to its crisp model by using Eq.16.

Step 2: Create the decision set which include the highest degree of truth-membership and the least degree of falsity and indeterminacy memberships.

Step 3: At the first node let the solution of linear programming model with integer restriction as an upper bound and the rounded-down integer solution as a lower bound.

Step 4: For branching process, we select the variable with the largest fractional part. Two constrains are obtained after the branching process, one for$\leq$ and the other is $\geq$ constraint.

Step 5: Create two nodes for the two new constraints.

Step 6: Solve the model again, after adding new constraints at each node.

Step 7: The optimal integer solution has been reached, if the feasible integer solution has the largest upper bound value of any ending node. Otherwise return to step 4.

The previous algorithm is for a maximization model. For a minimization model, the solution of linear programming problem with integer restrictions are rounded up and upper and lower bounds are reversed.

6 Numerical Examples

To measure the efficiency of our proposed model we solved many numerical examples.

6.1 Illustrative Example #1

$$
\begin{aligned}
&max \quad \tilde{5}x_1 + \tilde{3}x_2 \\
&subject\ to \\
&\qquad \tilde{4}x_1 + \tilde{3}x_2 \leq \widetilde{12} \\
&\qquad \tilde{1}x_1 + \tilde{3}x_2 \leq \tilde{6} \\
&\qquad x_1, x_2 \geq 0 \ and \ integer \\
&\quad where \\
&\quad \tilde{5} = \langle (4,5,6), 0.8, 0.6, 0.4 \rangle \\
&\quad \tilde{3} = \langle (2.5,3,3.5), 0.75, 0.5, 0.3 \rangle \\
&\quad \tilde{4} = \langle (3.5,4,4.1), 1, 0.5, 0.0 \rangle \\
&\quad \tilde{3} = \langle (2.5,3,3.5), 0.75, 0.5, 0.25 \rangle \\
&\quad \tilde{1} = \langle (0,1,2), 1, 0.5, 0 \rangle
\end{aligned}
$$

$\tilde{3} = \langle(2.8,3,3.2\ ), 0.75, 0.5, 0.25\ \rangle$
$\widetilde{12} = \langle(11,12,13\ ), 1, 0.5, 0\ \rangle$
$\tilde{6} = \langle(5.5,6,7.5\ ), 0.8, 0.6, 0.4\ \rangle$

Then the neutrosophic model converted to the crisp model by using Eq.16.as follows:

$max \quad 5.6875x_1 + 3.5968x_2$
subject to
$4.3125x_1 + 3.625x_2 \leq 14.375$
$0.2815x_1 + 3.925x_2 \leq 7.6375$
$x_1, x_2 \geq 0\ and\ integer$

The optimal solution of the problem is $x^* = (3,0)$ with optimal objective value 17.06250.

6.2 Illustrative Example #2

$max \quad \widetilde{25}x_1 + \widetilde{48}x_2$
$subject\ to$
$15x_1 + 30x_2 \leq 45000$
$24x_1 + 6x_2 \leq 24000$
$21x_1 + 14x_2 \leq 28000$
$x_1, x_2 \geq 0\ and\ integer$
where
$\widetilde{25} = \langle(19,25,33\ ), 0.8,0.5,0\ \rangle$;
$\widetilde{48} = \langle(44,48,54\ ), 0.9,0.5,0\ \rangle$

Then the neutrosophic model converted to the crisp model as:

max $27.8875x1 + 55.3x_2$
subject to
$15x_1 + 30x_2 \leq 45000$
$24x_1 + 6x_2 \leq 24000$
$21x_1 + 14x_2 \leq 28000$
$x_1, x_2 \geq 0\ and\ integer$

The optimal solution of the problem is $x^* = (500,1250)$ with optimal objective value 83068.75.

6.3 Illustrative Example #3

The owner of a machine shop is planning to expand by purchasing some new machines - presses and lathes. The owner has estimated that each press purchased will increase profit by $100 per day and each lathe will increase profit by $150 daily.

The number of machines the owner can purchase is limited by the cost of the machines and the available floor space in the shop. The machine purchase prices and space requirements are as follows.

Table 1. Requirements of machines

Machine	Required Floor Space (ft^2)	Purchase Price
Press	$\widetilde{40}$	\$8,000
Lathe	$\widetilde{70}$	\$4,000

The owner has a budget of \$40,000 for purchasing machines and 200 square feet of available floor space. The owner wants to know how many of each type of machine to purchase to maximize the daily increase in profit.

The problem can be formulated as follows:

max $100x_1 + 150x_2$

Subject to

$8{,}000x_1 + 4{,}000x_2 \leq 40{,}000$

$\widetilde{40}x_1 + \widetilde{70}x_2 \leq 40{,}000$

$x_1, x_2 \geq 0 \; and \; integer$

Since $\widetilde{40}$= {(30, 40, 50);(0.7, 0.4, 0.3)}

Since $\widetilde{70}$= {(50, 70, 120);(0.7, 0.4, 0.3)}

By using Neutrosophic Branch and Bound Algorithm, then by converting neutrosophic integer programming parameter to its crisp values by using Eq.16 then,

max $100x_1 + 150x_2$

Subject to

$8{,}000x_1 + 4{,}000x_2 \leq 40{,}000$

$15x_1 + 30x_2 \leq 40{,}000$

$x_1, x_2 \geq 0 \; and \; integer$

We began the branch and bound method by first solving the problem as a regular linear programming model without integer restrictions, the result as follows:

$x_1 = 2.22, x_2 = 5.56$, And optimal objective value = 1,055.56.

By applying branch and bound steps then, the upper and lower bounds at each node presented in Fig.2:

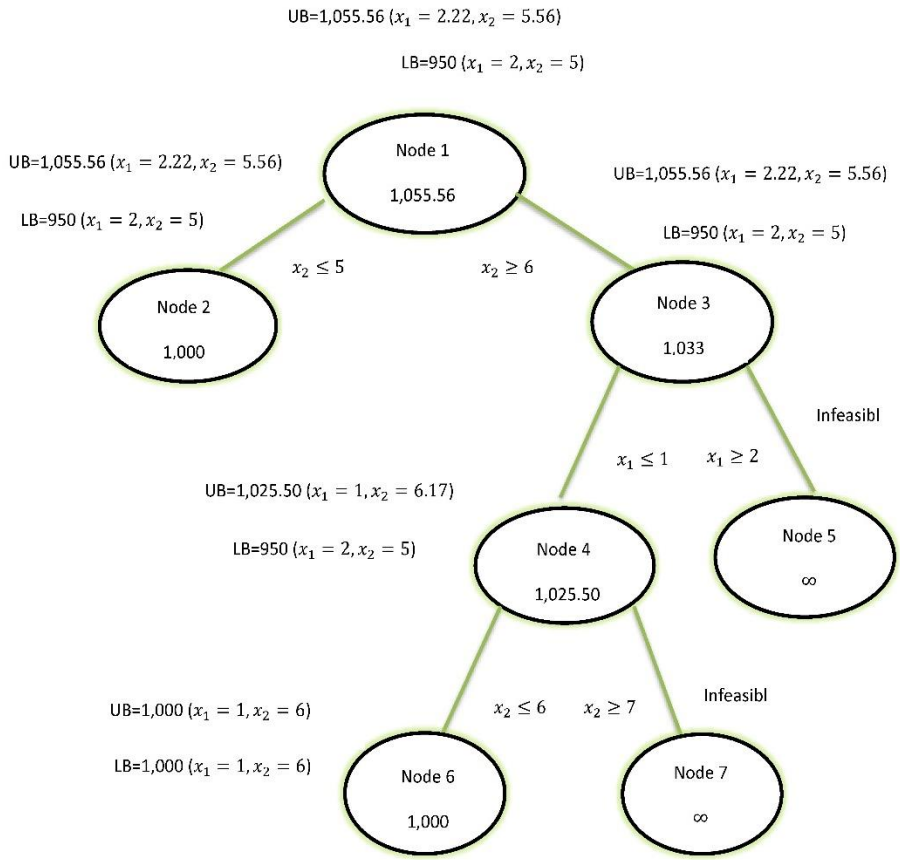

Fig.2. Branch and bound diagram with optimal solution at node 6

The previous branch and bound diagram indicates that the optimal integer solution $x_1 = 1, x_2 = 6$, has been reached at node 6 with optimal value =1000.

7 Conclusions and Future Work

In this chapter, we proposed an integer programming model based on neutrosophic environment, simultaneously considering the degrees of acceptance, indeterminacy and rejection of objectives, by proposed model for solving neutrosophic integer programming problems (NIPP). In the proposed model, we maximized the degrees of acceptance and minimized indeterminacy and rejection of objectives. NIPP was transformed into a crisp programming model using truth membership, indeterminacy membership, falsity membership and score functions. We also gave numerical examples to show the efficiency of the proposed method. As far as future directions are concerned, these will include studying the duality theory of integer programming problems based on Neutrosophics.

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IV

Neutrosophic Goal Programming

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Abstract

In this chapter, the goal programming in neutrosophic environment is introduced. The degree of acceptance, indeterminacy and rejection of objectives is considered simultaneous. In the two proposed models to solve Neutrosophic Goal Programming Problem (NGPP), our goal is to minimize the sum of the deviation in the model (I), while in the model (II), the neutrosophic goal programming problem NGPP is transformed into the crisp programming model using truth membership, indeterminacy membership, and falsity membership functions. Finally, the industrial design problem is given to illustrate the efficiency of the proposed models. The obtained results of Model (I) and Model (II) are compared with other methods.

Keywords

Neutrosophic optimization; Goal programming problem.

1 Introduction

Goal programming (GP) Models was originally introduced by Charnes and Cooper in early 1961 for a linear model . Multiple and conflicting goals can be used in goal programming. Also, GP allows the simultaneous solution of a system of Complex objectives, and the solution of the problem requires the establishment among these multiple objectives. In this case, the model must be solved in such a way that each of the objectives to be achieved. Therefore, the sum of the deviations from the ideal should be minimized in the objective function. It is important that measure deviations from the ideal should have a single scale, because deviations with different scales cannot be collected. However, the target value associated with each goal could be neutrosophic in the real-world application. In 1995, Smarandache [17] starting from philosophy (when [8]

fretted to distinguish between *absolute truth* and *relative truth* or between *absolute falsehood* and *relative falsehood* in logics, and respectively between *absolute membership* and *relative membership* or *absolute non-membership* and *relative non-membership* in set theory) [12] began to use the non-standard analysis. Also, inspired from the sport games (winning, defeating, or tie scores), from votes (pro, contra, null/black votes), from positive/negative/zero numbers, from yes/no/NA, from decision making and control theory (making a decision, not making, or hesitating), from accepted/rejected/pending, etc. and guided by the fact that the law of excluded middle did not work any longer in the modern logics. [12] combined the non-standard analysis with a tri-component logic/set/probability theory and with philosophy. How to deal with all of them at once, is it possible to unity them? [12].

Netrosophic theory means Neutrosophy applied in many fields in order to solve problems related to indeterminacy. Neutrosophy is a new branch of philosophy that studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra. This theory considers every entity <A> together with its opposite or negation <antiA> and with their spectrum of neutralities <neutA> in between them (i.e. entities supporting neither <A> nor<antiA>). The <neutA> and <antiA> ideas together are referred to as <nonA>.

Neutrosophy is a generalization of Hegel's dialectics (the last one is based on <A> and <antiA> only). According to this theory every entity <A> tends to be neutralized and balanced by <antiA> and <nonA> entities - as a state of equilibrium. In a classical way <A>, <neutA>, <antiA> are disjoint two by two. But, since in many cases the borders between notions are vague, imprecise, Sorites, it is possible that <A>, <neutA>, <antiA> (and <nonA> of course) have common parts two by two, or even all three of them as well. Hence, in one hand, the Neutrosophic Theory is based on the triad <A>, <neutA>, and <antiA>. In the other hand, Neutrosophic Theory studies the indeterminacy, labeled as I, with $I_n = I$ for $n \geq 1$, and $mI + nI = (m+n)I$, in neutrosophic structures developed in algebra, geometry, topology etc.

The most developed fields of Netrosophic theory are Neutrosophic Set, Neutrosophic Logic, Neutrosophic Probability, and Neutrosophic Statistics - that started in 1995, and recently Neutrosophic Precalculus and Neutrosophic Calculus, together with their applications in practice. Neutrosophic Set and Neutrosophic Logic are generalizations of the fuzzy set and respectively fuzzy logic (especially of intuitionistic fuzzy set and respectively intuitionistic fuzzy logic). In neutrosophic logic a proposition has a degree of truth (*T*), a degree of indeterminacy (*I*), and a degree of falsity (*F*), where *T,I,F* are standard or non-standard subsets of $]^{-}0, 1^{+}[$.

The important method for multi-objective decision making is goal programming approaches in practical decision making in real life. In a standard GP formulation, goals and constraints are defined precisely, but sometimes the system aim and conditions include some vague and undetermined situations. In particular, expressing the decision maker's unclear target levels for the goals mathematically and the need to optimize all goals at the same needs to complicated calculations.

The neutrosophic approach for goal programming tries to solve this kind of unclear difficulties in this chapter.

The organization of the chapter is as follows. The next section introduces a brief some preliminaries. Sections 3 describe the formation of the Problem and develop two models to neutrosophic goal programming. Section 4 presents an industrial design problem is provided to demonstrate how the approach can be applied. Finally, conclusions are provided in section 5.

2 Some Preliminaries

Definition 1. [17]

A real fuzzy number $\tilde{J}$ is a continuous fuzzy subset from the real line R whose triangular membership function $\mu_{\tilde{J}}(J)$ is defined by a continuous mapping from R to the closed interval [0,1], where

(1) $\mu_{\tilde{J}}(J)=0$ for all $J\in(-\infty,a_1]$,

(2) $\mu_{\tilde{J}}(J)$ is strictly increasing on $J\in[a_1,m]$,

(3) $\mu_{\tilde{J}}(J)=1$ for $J=m$,

(4) $\mu_{\tilde{J}}(J)$ is strictly decreasing on $J\in[m,a_2]$,

(5) $\mu_{\tilde{J}}(J)=0$ for all $J\in[a_2,+\infty)$.

This will be elicited by:

$$\mu_{\tilde{J}}(J)=\begin{cases}\dfrac{J-a_1}{m-a_1}, & a_1\le J\le m,\\ \dfrac{a_2-J}{a_2-m}, & m\le J\le a_2,\\ 0, & otherwise.\end{cases} \tag{1}$$

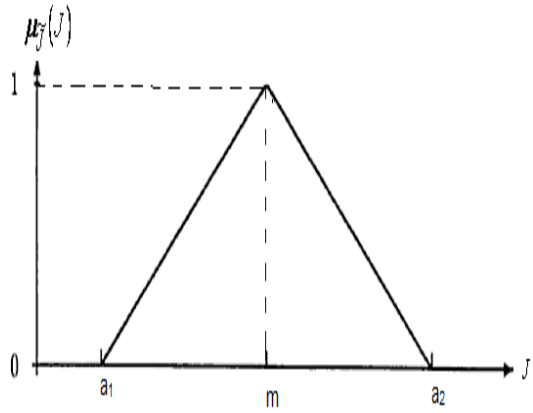

Fig. 1: Membership Function of Fuzzy Number *J*.

Where m is a given value a_1 and a_2 denote the lower and upper bounds. Sometimes, it is more convenient to use the notation explicitly highlighting the membership function parameters. In this case, we obtain

$$\mu(J;a_1,m,a_2)=\text{Max}\left\{\text{Min}\left[\frac{J-a_1}{m-a_1},\frac{a_2-J}{a_2-m}\right],0\right\} \tag{2}$$

In what follows, the definition of the α-level set or α-cut of the fuzzy number $\tilde{J}$ is introduced.

Definition 2. [1]

Let $X = \{x_1, x_2, \ldots, x_n\}$ be a fixed non-empty universe, an intuitionistic fuzzy set IFS *A* in *X* is defined as

$$A=\left\{\left\langle x,\mu_A(x),\upsilon_A(x)\right\rangle \middle| x\in X\right\} \tag{3}$$

which is characterized by a membership function $\mu_A : X \to [0,1]$ and a non-membership function $\upsilon_A : X \to [0,1]$ with the condition $0 \le \mu_A(x)+\upsilon_A(x) \le 1$ for all $x \in X$ where μ_A and υ_A represent, respectively, the degree of membership and non-membership of the element *x* to the set *A*. In addition, for each IFS *A* in *X*, $\pi_A(x)=1-\mu_A(x)-\upsilon_A(x)$ for all $x \in X$ is called the degree of hesitation of the element *x* to the set *A*. Especially, if $\pi_A(x)=0,$ then the IFS *A* is degraded to a fuzzy set.

Definition 3. [4] The α-level set of the fuzzy parameters $\tilde{J}$ in problem (1) is defined as the ordinary set $L_\alpha\left(\tilde{J}\right)$ for which the degree of membership function exceeds the level, α, $\alpha \in [0,1]$, where:

$$L_\alpha\left(\tilde{J}\right)=\left\{J \in R \,\middle|\, \mu_{\tilde{J}}\left(J\right)\geq \alpha\right\} \tag{4}$$

For certain values α_j^* to be in the unit interval.

Definition 4. [10] Let X be a space of points (objects) and $x \in X$. A neutrosophic set A in X is defined by a truth-membership function (x), an indeterminacy-membership function (x) and a falsity-membership function (x). It has been shown in figure 2. (x), (x) and (x) are real standard or real nonstandard subsets of]0−,1+[. That is $T_A(x):X\rightarrow$]0−,1+[, $I_A(x):X\rightarrow$]0−,1+[and $F_A(x):X\rightarrow$]0−,1+[. There is not restriction on the sum of (x), (x) and (x), so $0^-\leq \sup T_A(x)\leq \sup I_A(x)\leq F_A(x)\leq 3^+$.

In the following, we adopt the notations $\mu(x)$, $\sigma_A(x)$ and $v_A(x)$ instead of $T_A(x)$, $I_A(x)$ and $F_A(x)$, respectively. Also, we write SVN numbers instead of single valued neutrosophic numbers.

Definition 5. [10] Let X be a universe of discourse. A single valued neutrosophic set A over X is an object having the form

$$A=\{\langle x, \mu_A(x), \sigma_A(x), v_A(x)\rangle : x \in X\},$$

where $\mu_A(x):X\rightarrow[0,1]$, $\sigma_A(x):X\rightarrow[0,1]$ and $v_A(x):X\rightarrow[0,1]$ with $0\leq \mu_A(x)+\sigma_A(x)+v_A(x)\leq 3$ for all $x \in X$. The intervals $\mu(x)$, $\sigma_A(x)$ and $v_A(x)$ denote the truth-membership degree, the indeterminacy-membership degree and the falsity membership degree of x to A, respectively.

For convenience, a SVN number is denoted by $A=(a,b,c)$, where $a,b,c \in [0,1]$ and $a+b+c\leq 3$.

Definition 6. Let $\tilde{J}$ be a neutrosophic number in the set of real numbers R, then its truth-membership function is defined as

$$T_{\tilde{J}}\left(J\right)=\begin{cases}\dfrac{J-a_1}{a_2-a_1}, & a_1\leq J\leq a_2,\\ \dfrac{a_2-J}{a_3-a_2}, & a_2\leq J\leq a_3,\\ 0, & otherwise.\end{cases} \tag{5}$$

its indeterminacy-membership function is defined as

$$I_{\tilde{J}}(J)=\begin{cases}\dfrac{J-b_1}{b_2-b_1}, & b_1\le J\le b_2,\\ \dfrac{b_2-J}{b_3-b_2}, & b_2\le J\le b_3,\\ 0, & otherwise.\end{cases} \tag{6}$$

and its falsity-membership function is defined as

$$F_{\tilde{J}}(J)=\begin{cases}\dfrac{J-c_1}{c_2-c_1}, & c_1\le J\le c_2,\\ \dfrac{c_2-J}{c_3-c_2}, & c_2\le J\le c_3,\\ 1, & otherwise.\end{cases} \tag{7}$$

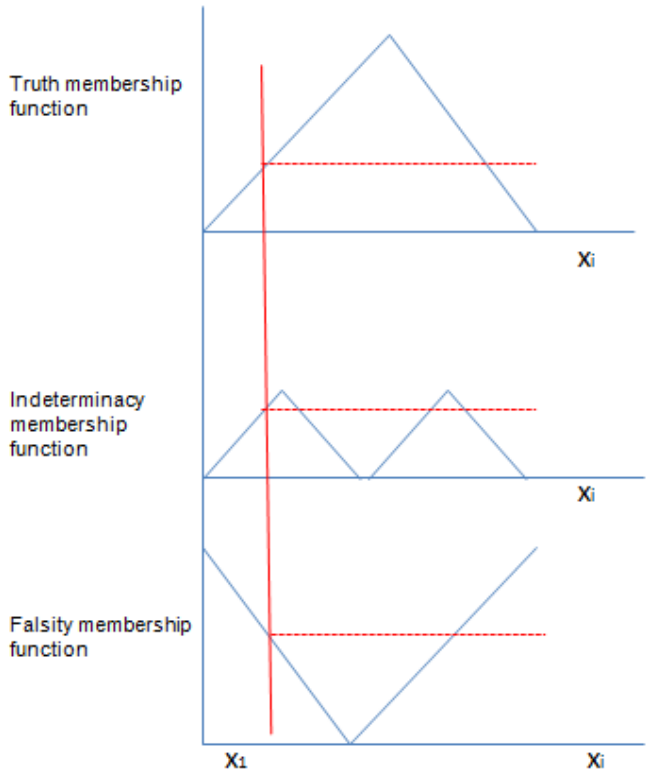

Fig. 2: Neutrosophication process [11]

3 Neutrosophic Goal Programming Problem

Goal programming can be written as:

Find $x = (x_1, x_2, ..., x_n)^T$

To achieve:

$$z_i = t_i, \quad i = 1, 2, ..., k \tag{8}$$

Subject to

$x \in X$

where t_i, are scalars and represent the target achievement levels of the objective functions that the decision maker wishes to attain provided, X is feasible set of the constraints.

The achievement function of the (8) model is the following:

$$Min \sum_{i=1}^{k} (w_{1i} n_i + w_{2i} p_i) \tag{9}$$

Goal and constraints:

$$z_i + n_i - p_i = t_i, \; i \in \{1, 2, ..., k\}$$

$$x \in X, \; n, p \geq 0, \; n \cdot p = 0$$

n_i, p_i are negative and positive deviations from t_i target.

The NGPP can be written as:

Find $x = (x_1, x_2, ..., x_n)^T$

So as to:

Minimize z_i with target value t_i, acceptance tolerance

a_i, indeterminacy tolerance d_i, rejection tolerance c_i,

Subject to

$x \in X$

$g_j(x) \leq b_j, \; j = 1, 2, ..., m$

$x_i \geq 0, \qquad i = 1, 2, ..., n$

with truth-membership, indeterminacy-membership and falsity-membership functions:

$$\mu_i^I(z_i)=\begin{cases}1, & if \quad z_i \le t_i,\\ 1-\dfrac{z_i-t_i}{a_i}, & if \quad t_i \le z_i \le t_i+a_i,\\ 0, & if \quad z_i \ge t_i+a_r\end{cases} \tag{10}$$

$$\sigma_i^I(z_i)=\begin{cases}0, & if \quad z_i \le t_i,\\ \dfrac{z_i-t_i}{d_i}, & if \quad t_i \le z_i \le t_i+d_i,\\ 1-\dfrac{z_i-t_i}{a_i-d_i}, & if \quad t_i+d_i \le z_i \le t_i+a_i,\\ 0, & if \quad z_i \ge t_i+a_i\end{cases} \tag{11}$$

$$\upsilon_i^I(z_i)=\begin{cases}0, & if \quad z_i \le t_i,\\ \dfrac{z_i-t_i}{C_i}, & if \quad t_i \le z_i \le t_i+C_i,\\ 1, & if \quad z_i \ge t_i+C_i\end{cases} \tag{12}$$

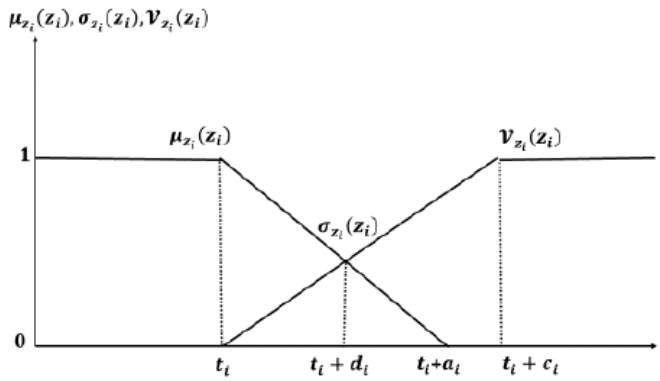

Fig. 3: Truth-membership, indeterminacy-membership and falsity-membership functions for z_i.

To maximize the degree the accptance and indeterminacy of NGP objectives and constriants, also to minimize the dgree of rejection of NGP objectives and constriants

$$\begin{aligned} &Max \ \ \mu_{z_i}(z_i), \ \ i=1,2,...,k\\ &Max \ \ \sigma_{z_i}(z_i), \ \ i=1,2,...,k\\ &Min \ \ \upsilon_{z_i}(z_i), \ \ i=1,2,...,k\end{aligned} \tag{13}$$

Subject to

$$0 \le \mu_{z_i}(z_i) + \sigma_{z_i}(z_i) + \upsilon_{z_i}(z_i) \le 3, \ i = 1,2,...,k$$

$$\upsilon_{z_i}(z_i) \ge 0, \ \ i = 1,2,...,k$$

$$\mu_{z_i}(z_i) \ge \upsilon_{z_i}(z_i), \ \ i = 1,2,...,k$$

$$\mu_{z_i}(z_i) \ge \sigma_{z_i}(z_i), \ \ i = 1,2,...,k$$

$$g_j(x) \le b_j, \ j = 1,2,...,m$$

$$x \in X$$

$$x_j \ge 0, \qquad j = 1,2,...,n$$

where $\mu_{z_i}(z_i), \ \sigma_{z_i}(z_i), \ \upsilon_{z_i}(z_i)$ are truth membership function, indeterminacy membership function, falsity membership function of Neutrosophic decision set respectively.

The highest degree of truth membership function is unity. So, for the defined the truth membership function $\mu_{z_i}(z_i)$, the flexible membership goals having the aspired level unity can be presented as

$$\mu_{z_i}(z_i) + n_{i1} - p_{i1} = 1$$

For case of indeterminacy (indeterminacy membership function), it can be written:

$$\sigma_{z_i}(z_i) + n_{i2} - p_{i2} = 0.5$$

For case of rejection (falsity membership function), it can be written

$$\mu_{z_i}(z_i) + n_{i3} - p_{i3} = 0$$

Here $n_{i1}, p_{i1}, \ n_{i2}, p_{i2}, \ n_{i3}$ *and* p_{i3} are under-deviational and over-deviational variables.

Our goals are maximize the degree of the accptance and indeterminacy of NGP objectives and constriants, and minimize the dgree of rejection of NGP objectives and constriants.

Model (I). The minimization of the sum of the deviation can be formulated as:

$$Min\ \lambda = \sum_{i=1}^{k} w_{i1} n_{i1} + \sum_{i=1}^{k} w_{i2} n_{i2} + \sum_{i=1}^{k} w_{i3} p_{i3} \qquad (14)$$

Subject to

$$\mu_{z_i}(z_i) + n_{i1} \ge 1, \ \ i = 1,2,...,k$$

$$\sigma_{z_i}(z_i)+n_{i2}\geq 0.5,\ i=1,2,...,k$$

$$\upsilon_{z_i}(z_i)-p_{i3}\leq 0,\ i=1,2,...,k$$

$$\upsilon_{z_i}(z_i)\geq 0,\ i=1,2,...,k$$

$$\mu_{z_i}(z_i)\geq \upsilon_{z_i}(z_i),\ i=1,2,...,k$$

$$\mu_{z_i}(z_i)\geq \sigma_{z_i}(z_i),\ i=1,2,...,k$$

$$0\leq \mu_{z_i}(z_i)+\sigma_{z_i}(z_i)+\upsilon_{z_i}(z_i)\leq 3,\ i=1,2,...,k$$

$$g_j(x)\leq b_j,\ j=1,2,...,m$$

$$n_{i1}, n_{i2},\ p_{i3}\geq 0,\ i=1,2,...,k$$

$$x\in X$$

$$x_j\geq 0,\qquad j=1,2,...,n$$

On the other hand, neutrosophic goal programming NGP in Model (13) can be represented by crisp programming model using truth membership, indeterminacy membership, and falsity membership functions as:

$$Max\ \alpha, Max\ \gamma, Min\ \beta \tag{15}$$

$$\mu_{z_i}(z_i)\geq \alpha,\ i=1,2,...,k$$

$$\sigma_{z_i}(z_i)\geq \gamma,\ i=1,2,...,k$$

$$\upsilon_{z_i}(z_i)\leq \beta,\ i=1,2,...,k$$

$$z_i\leq t_i,\ i=1,2,...,k$$

$$0\leq \alpha+\gamma+\beta\leq 3$$

$$\alpha,\gamma\geq 0,\ \beta\leq 1$$

$$g_j(x)\leq b_j,\ j=1,2,...,m$$

$$x_j\geq 0,\qquad j=1,2,...,n$$

In model (15) the $Max\ \alpha, Max\ \gamma$ are equivalent to $Min(1-\alpha), Min(1-\gamma)$ respectively where $0\leq\alpha,\ \gamma\leq 1$

$$Min\ \beta(1-\alpha)(1-\gamma) \tag{16}$$

Subject to

$$z_i\leq t_i+a_i(a_i-d_i)\beta(1-\alpha)(1-\gamma),\ i=1,2,...,k$$

$$z_i\leq t_i,\ i=1,2,...,k$$

$$0 \le \alpha + \gamma + \beta \le 3$$

$$\alpha, \gamma \ge 0, \ \beta \le 1$$

$$g_j(x) \le b_j, \ j = 1, 2, ..., m$$

$$x_j \ge 0, \qquad j = 1, 2, ..., n$$

If we take $\beta(1-\alpha)(1-\gamma) = v$ the model (16) becomes:

Model (II).

$$\text{Minimize } v \tag{17}$$

Subject to

$$z_i \le t_i + a_i(a_i - d_i)v, \ i = 1, 2, ..., k$$

$$z_i \le t_i, \ i = 1, 2, ..., k$$

$$0 \le \alpha + \gamma + \beta \le 3$$

$$\alpha, \gamma \ge 0, \ \beta \le 1$$

$$g_j(x) \le b_j, \ j = 1, 2, ..., m$$

$$x_j \ge 0, \qquad j = 1, 2, ..., n$$

The crisp model (17) is solved by using any mathematical programming technique with *v* as parameter to get optimal solution of objective functions.

4 Illustrative Example

This industrial application selected from [15]. Let the Decision maker wants to remove about 98.5% biological oxygen demand (BOD) and the tolerances of acceptance, indeterminacy and rejection on this goal are 0.1, 0.2 and 0.3 respectively. Also, Decision maker wants to remove the said amount of $BODS_5$ within 300 (thousand $) tolerances of acceptance, indeterminacy and rejection 200, 250, 300 (thousand $) respectively. Then the neutrosophic goal programming problem is:

$$\min z_1(x_1, x_2, x_3, x_4) = 19.4x_1^{-1.47} + 16.8x_2^{-1.66} + 91.5x_3^{-0.3} + 120x_4^{-0.33},$$

$$\min z_2(x_1, x_2, x_3, x_4) = x_1 x_2 x_3 x_4,$$

$$s.t.:$$

$$x_i \ge 0, \ i = 1, 2, 3, 4.$$

With target 300, acceptance tolerance 200, indeterminacy tolerance 100 , and rejection tolerance 300 for the first objective z_1.

Also, with target 0.015, acceptance tolerance 0.1, indeterminacy tolerance 0.05, and rejection tolerance 0.2 for the second objective z_2.

Where x_i is the percentage BOD5(to remove 5 days BOD) after each step. Then after four processes the remaining percentage of BOD5 will be x_i, *i=1, 2, 3, 4*. The aim is to minimize the remaining percentage of BOD5 with minimum annual cost as much as possible. The annual cost of BOD5 removal by various treatments is primary clarifier, trickling filter, activated sludge, carbon adsorption. z_1 represent the annual cost. While z_2 represent removed from the wastewater.

The truth membership, indeterminacy membership, falsity membership functions were considered to be neutrosophic triangular.

The truth membership functions of the goals are obtained as follows:

$$\mu_1^I(z_1)=\begin{cases}1, & if \quad z_1 \le 300,\\ 1-\dfrac{z_1-300}{200}, & if \quad 300 \le z_1 \le 500,\\ 0, & if \quad z_1 \ge 500\end{cases}$$

$$\mu_2^I(z_2)=\begin{cases}1, & if \quad z_2 \le 0.015,\\ 1-\dfrac{z_2-0.015}{0.1}, & if \quad 0.015 \le z_2 \le 0.115,\\ 0, & if \quad z_2 \ge 0.115.\end{cases}$$

The indeterminacy membership functions of the goals are given:

$$\sigma_1^I(z_1)=\begin{cases}0, & if \quad z_1 \le 300,\\ \dfrac{z_1-300}{100}, & if \quad 300 \le z_1 \le 400,\\ 1-\dfrac{z_1-300}{100}, & if \quad 400 \le z_1 \le 600,\\ 0, & if \quad z_1 \ge 600\end{cases}$$

$$\sigma_2^I(z_2)=\begin{cases}0, & if \quad z_2 \le 0.015,\\ \dfrac{z_2-0.015}{0.05}, & if \quad 0.015 \le z_2 \le 0.065,\\ 1-\dfrac{z_2-0.015}{0.05}, & if \quad 0.065 \le z_2 \le 0.215,\\ 0, & if \quad z_2 \ge 0.215\end{cases}$$

The falsity membership functions of the goals are obtained as follows:

$$\upsilon_1^I(z_1)=\begin{cases}0, & if \quad z_1 \le 300,\\ \dfrac{z_1-300}{300}, & if \quad 300 \le z_1 \le 600,\\ 1, & if \quad z_1 \ge 600\end{cases}$$

$$\upsilon_2^I(z_2)=\begin{cases}0, & if \quad z_2 \le 0.015,\\ \dfrac{z_2-0.015}{0.2}, & if \quad 0.015 \le z_2 \le 0.215,\\ 1, & if \quad z_2 \ge 0.215\end{cases}$$

The software LINGO 15.0 is used to solve this problem. Table (1) shows the comparison of the obtained results among the proposed models and the others methods.

Table 1: Comparison of optimal solution based on different methods:

Methods	z_1	z_2	x_1	x_2	x_3	x_4
FG^2P^2 Ref[15]	363.8048	0.04692	0.705955	0.7248393	0.1598653	0.5733523
IFG^2P^2 Ref[15]	422.1483	0.01504	0.638019	0.662717	0.09737155	0.3653206
Model (I)	317.666	0.1323	0.774182	0.7865418	0.2512332	0.8647621
Model (II)	417.6666	0.2150	2.628853	3.087266	0.181976E-01	1.455760

It is to be noted that model (I) offers better solutions than other methods.

5 Conclusions and Future Work

The main purpose of this chapter was to introduce goal programming in neutrosophic environment. The degree of acceptance, indeterminacy and rejection of objectives are considered simultaneously. Two proposed models to solve neutrosophic goal programming problem (NGPP), in the first model, our goal is to minimize the sum of the deviation, while the second model, neutrosophic goal programming NGP is transformed into crisp programming model using truth membership, indeterminacy membership, and falsity membership functions.

Finally, a numerical experiment is given to illustrate the efficiency of the proposed methods.

Moreover, the comparative study has been held of the obtained results and has been discussed. In the future studies, the proposed algorithm can be solved by metaheuristic algorithms.

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V

Taylor Series Approximation to Solve Neutrosophic Multi-objective Programming Problem

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Abstract

In this chapter, Taylor series is used to solve neutrosophic multi-objective programming problem (NMOPP). In the proposed approach, the truth membership, indeterminacy membership, falsity membership functions associated with each objective of multi-objective programming problems are transformed into a single objective linear programming problem by using a first order Taylor polynomial series. Finally, to illustrate the efficiency of the proposed method, a numerical experiment for supplier selection is given as an application of Taylor series method for solving neutrosophic multi-objective programming problem.

Keywords

Taylor series; Neutrosophic optimization; Multi-objective programming problem.

1 Introduction

In 1995, Smarandache [13], starting from philosophy (when [8] fretted to distinguish between absolute truth and relative truth or between absolute falsehood and relative falsehood in logics, and respectively between absolute membership and relative membership or absolute non-membership and relative non-membership in set theory) [12], began to use the non-standard analysis. Also, inspired from the sport games (winning, defeating, or tie scores), from votes (pro, contra, null/black votes), from positive/negative/zero numbers, from yes/no/NA,

from decision making and control theory (making a decision, not making one, or hesitating), from accepted/rejected/pending, etc., and guided by the fact that the law of excluded middle did not work any longer in the modern logics [12], Smarandache combined the non-standard analysis with a tri-component logic/set/probability theory and with philosophy. How to deal with all of them at once, is it possible to unify them? [12].

Netrosophic theory means Neutrosophy applied in many fields in order to solve problems related to indeterminacy. Neutrosophy is a new branch of philosophy that studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra. This theory considers every entity <A> together with its opposite or negation <antiA> and with their spectrum of neutralities <neutA> in between them (i.e. entities supporting neither <A> nor<antiA>). The <neutA> and <antiA> ideas together are referred to as <nonA>.

Neutrosophy is a generalization of Hegel's dialectics (the last one is based on <A> and <antiA> only). According to this theory every entity <A> tends to be neutralized and balanced by <antiA> and <nonA> entities - as a state of equilibrium. In a classical way <A>, <neutA>, <antiA> are disjoint two by two. But, since in many cases the borders between notions are vague, imprecise, Sorites, it is possible that <A>, <neutA>, <antiA> (and <nonA> of course) have common parts two by two, or even all three of them as well. Hence, in one hand, the Neutrosophic Theory is based on the triad <A>, <neutA>, and <antiA>. In the other hand, Neutrosophic Theory studies the indeterminacy, labeled as I, with $I_n = I$ for $n \geq 1$, and $mI + nI = (m+n)I$, in neutrosophic structures developed in algebra, geometry, topology etc.

The most developed fields of Netrosophic theory are Neutrosophic Set, Neutrosophic Logic, Neutrosophic Probability, and Neutrosophic Statistics - that started in 1995, and recently Neutrosophic Precalculus and Neutrosophic Calculus, together with their applications in practice. Neutrosophic Set and Neutrosophic Logic are generalizations of the fuzzy set and respectively fuzzy logic (especially of intuitionistic fuzzy set and respectively intuitionistic fuzzy logic). In neutrosophic logic a proposition has a degree of truth (*T*), a degree of indeterminacy (*I*), and a degree of falsity (*F*), where *T, I, F* are standard or non-standard subsets of $]^{-}0, 1^{+}[$.

Multi-objective linear programming problem (MOLPP) a prominent tool for solving many real decision-making problems like game theory, inventory problems, agriculture based management systems, financial and corporate planning, production planning, marketing and media selection, university planning and student admission, health care and hospital planning, air force maintenance units, bank branches etc.

Our objective in this chapter is to propose an algorithm to the solution of neutrosophic multi-objective programming problem (NMOPP) with the help of the first order Taylor's theorem. Thus, neutrosophic multi-objective linear programming problem is reduced to an equivalent multi-objective linear programming problem. An algorithm is proposed to determine a global optimum to the problem in a finite number of steps. The feasible region is a bounded set. In the proposed approach, we have attempted to reduce computational complexity in the solution of (NMOPP). The proposed algorithm is applied to supplier selection problem.

The rest of this chapter is organized as follows. Section 2 gives brief some preliminaries. Section 3 describes the formation of the problem. Section 4 presents the implementation and validation of the algorithm with practical application. Finally, section 6 presents the conclusion and proposals for future work.

2 Some preliminaries

Definition 1. [1] A triangular fuzzy number $\tilde{J}$ is a continuous fuzzy subset from the real line R whose triangular membership function $\mu_{\tilde{J}}(J)$ is defined by a continuous mapping from R to the closed interval [0,1], where

(6) $\mu_{\tilde{J}}(J) = 0$ for all $J \in (-\infty, a_1]$,

(7) $\mu_{\tilde{J}}(J)$ is strictly increasing on $J \in [a_1, m]$,

(8) $\mu_{\tilde{J}}(J) = 1$ for $J = m$,

(9) $\mu_{\tilde{J}}(J)$ is strictly decreasing on $J \in [m, a_2]$,

(10) $\mu_{\tilde{J}}(J) = 0$ for all $J \in [a_2, +\infty)$.

This will be elicited by:

$$\mu_{\tilde{J}}(J) = \begin{cases} \dfrac{J - a_1}{m - a_1}, & a_1 \le J \le m, \\ \dfrac{a_2 - J}{a_2 - m}, & m \le J \le a_2, \\ 0, & otherwise. \end{cases} \tag{18}$$

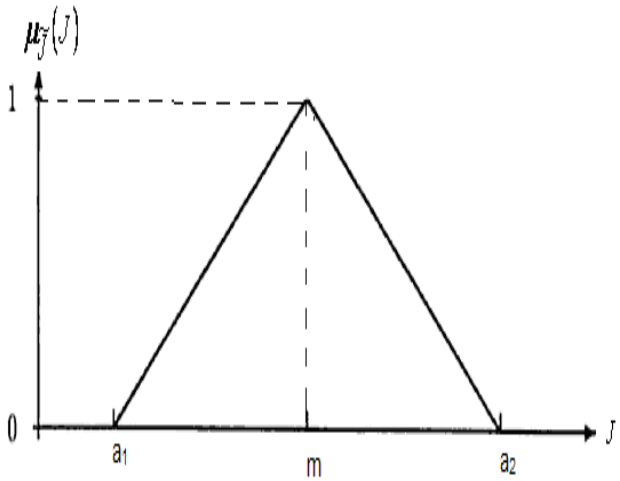

Fig. 3: Membership Function of Fuzzy Number *J*.

where *m* is a given value a_1 & a_2 denoting the lower and upper bounds. Sometimes, it is more convenient to use the notation explicitly highlighting the membership function parameters. In this case, we obtain

$$\mu(J;a_1,m,a_2)=\text{Max}\left\{\text{Min}\left[\frac{J-a_1}{m-a_1},\frac{a_2-J}{a_2-m}\right],0\right\} \tag{19}$$

In what follows, the definition of the α-level set or α-cut of the fuzzy number $\tilde{J}$ is introduced.

Definition 2. [1] Let $X = \{x_1, x_2, \ldots, x_n\}$ be a fixed non-empty universe. An intuitionistic fuzzy set IFS *A* in *X* is defined as

$$A=\left\{\left\langle x,\mu_A(x),\upsilon_A(x)\right\rangle \middle| x\in X\right\} \tag{20}$$

which is characterized by a membership function $\mu_A : X \to [0,1]$ and a non-membership function $\upsilon_A : X \to [0,1]$ with the condition $0 \le \mu_A(x)+\upsilon_A(x) \le 1$ for all $x \in X$ where μ_A and υ_A represent ,respectively, the degree of membership and non-membership of the element *x* to the set *A*. In addition, for each IFS *A* in *X* , $\pi_A(x)=1-\mu_A(x)-\upsilon_A(x)$ for all $x \in X$ is called the degree of hesitation of the element *x* to the set *A* . Especially, if $\pi_A(x)=0,$, then the IFS *A* is degraded to a fuzzy set.

Definition 3. [4] The α-level set of the fuzzy parameters $\tilde{J}$ in problem (1) is defined as the ordinary set $L_\alpha\left(\tilde{J}\right)$ for which the degree of membership function exceeds the level, α, $\alpha \in [0,1]$, where:

$$L_\alpha\left(\tilde{J}\right)=\left\{J \in R \left| \mu_{\tilde{J}}\left(J\right) \geq \alpha\right.\right\} \tag{21}$$

For certain values α_j^* to be in the unit interval.

Definition 4. [10] Let X be a space of points (objects) and $x \in X$. A neutrosophic set A in X is defined by a truth-membership function (x), an indeterminacy-membership function (x) and a falsity-membership function $F(x)$. It has been shown in figure 2. (x), (x) and $F(x)$ are real standard or real nonstandard subsets of]0−,1+[. That is $T_A(x):X \rightarrow]0-,1+[$, $I_A(x):X \rightarrow]0-,1+[$ and $F_A(x):X \rightarrow]0-,1+[$. There is not restriction on the sum of (x), (x) and $F(x)$, so $0- \leq \sup T_A(x) \leq \sup I_A(x) \leq F_A(x) \leq 3+$.

In the following, we adopt the notations $\mu(x)$, $\sigma_A(x)$ and $v_A(x)$ instead of $T_A(x)$, $I_A(x)$ and $F_A(x)$, respectively. Also, we write SVN numbers instead of single valued neutrosophic numbers.

Definition 5. [10] Let X be a universe of discourse. A single valued neutrosophic set A over X is an object having the form

$$A=\{\langle x, \mu_A(x), \sigma_A(x), v_A(x)\rangle : x \in X\},$$

where $\mu_A(x):X \rightarrow [0,1]$, $\sigma_A(x):X \rightarrow [0,1]$ and $v_A(x):X \rightarrow [0,1]$ with $0 \leq \mu_A(x)+\sigma_A(x)+v_A(x) \leq 3$ for all $x \in X$. The intervals $\mu(x)$, $\sigma_A(x)$ and $v_A(x)$ denote the truth-membership degree, the indeterminacy-membership degree and the falsity membership degree of x to A, respectively.

For convenience, a SVN number is denoted by $A=(a,b,c)$, where $a,b,c \in [0,1]$ and $a+b+c \leq 3$.

Definition 6 Let $\tilde{J}$ be a neutrosophic triangular number in the set of real numbers R, then its truth-membership function is defined as

$$T_{\tilde{J}}\left(J\right)=\begin{cases}\dfrac{J-a_1}{a_2-a_1}, & a_1 \leq J \leq a_2,\\ \dfrac{a_2-J}{a_3-a_2}, & a_2 \leq J \leq a_3,\\ 0, & otherwise.\end{cases} \tag{22}$$

its indeterminacy-membership function is defined as

$$I_{\tilde{J}}(J)=\begin{cases}\dfrac{J-b_1}{b_2-b_1}, & b_1 \le J \le b_2,\\ \dfrac{b_2-J}{b_3-b_2}, & b_2 \le J \le b_3,\\ 0, & otherwise.\end{cases} \tag{23}$$

and its falsity-membership function is defined as

$$F_{\tilde{J}}(J)=\begin{cases}\dfrac{J-c_1}{c_2-c_1}, & c_1 \le J \le c_2,\\ \dfrac{c_2-J}{c_3-c_2}, & c_2 \le J \le c_3,\\ 1, & otherwise.\end{cases} \tag{24}$$

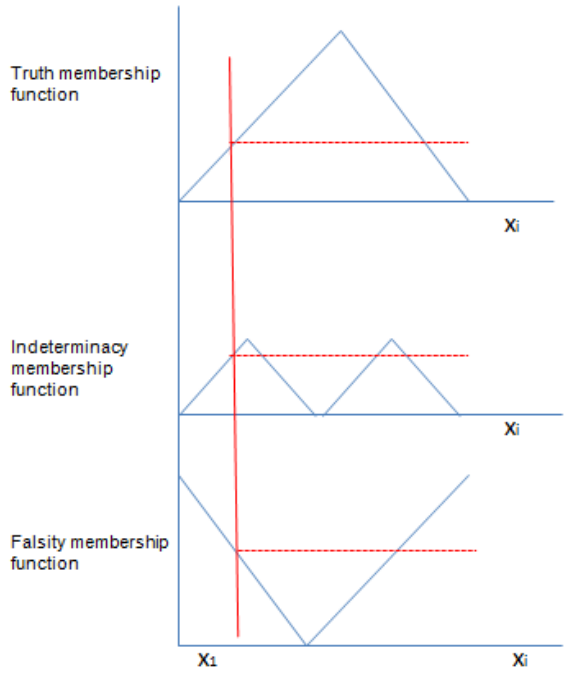

Fig. 4: Neutrosophication process [11]

3 Formation of The Problem

The multi-objective linear programming problem and the multi- objective neutrosophic linear programming problem are described in this section.

A. Multi-objective Programming Problem (MOPP)

In this chapter, the general mathematical model of the MOPP is as follows [6]:

$$\min/\max \left[z_1(x_1,...,x_n), z_2(x_1,...,x_n),...,z_p(x_1,...,x_n) \right] \tag{8}$$

$$subject\ to\ \ x \in S, x \geq 0$$

$$S = \left\{ x \in R^n \left| AX \begin{pmatrix} \leq \\ = \\ \geq \end{pmatrix} b, \quad X \geq 0. \right. \right\} \tag{25}$$

B. Neutrosophic Multi-objective Programming Problem (NMOPP)

If an imprecise aspiration level is introduced to each of the objectives of MOPP, then these neutrosophic objectives are termed as neutrosophic goals.

Let $z_i \in \left[z_i^L, z_i^U \right]$ denote the imprecise lower and upper bounds respectively for the i^{th} neutrosophic objective function.

For maximizing objective function, the truth membership, indeterminacy membership, falsity membership functions can be expressed as follows:

$$\mu_i^I(z_i) = \begin{cases} 1, & if \quad z_i \geq z_i^U, \\ \dfrac{z_i - z_i^L}{z_i^U - z_i^L}, & if \quad z_i^L \leq z_i \leq z_i^U, \\ 0, & if \quad z_i \leq z_i^L \end{cases} \tag{26}$$

$$\sigma_i^I(z_i) = \begin{cases} 1, & if \quad z_i \geq z_i^U, \\ \dfrac{z_i - z_i^L}{z_i^U - z_i^L}, & if \quad z_i^L \leq z_i \leq z_i^U, \\ 0, & if \quad z_i \leq z_i^L \end{cases} \tag{27}$$

$$\upsilon_i^I(z_i) = \begin{cases} 0, & if \quad z_i \geq z_i^U, \\ \dfrac{z_i - z_i^L}{z_i^U - z_i^L}, & if \quad z_i^L \leq z_i \leq z_i^U, \\ 1, & if \quad z_i \leq z_i^L \end{cases} \tag{28}$$

For minimizing objective function, the truth membership, indeterminacy membership, falsity membership functions can be expressed as follows:

$$\mu_i^I(z_i)=\begin{cases}1, & if \quad z_i \le z_i^L,\\ \dfrac{z_i^U - z_i}{z_i^U - z_i^L}, & if \quad z_i^L \le z_i \le z_i^U,\\ 0, & if \quad z_i \ge z_i^U\end{cases} \tag{29}$$

$$\sigma_i^I(z_i)=\begin{cases}1, & if \quad z_i \le z_i^L,\\ \dfrac{z_i^U - z_i}{z_i^U - z_i^L}, & if \quad z_i^L \le z_i \le z_i^U,\\ 0, & if \quad z_i \ge z_i^U\end{cases} \tag{30}$$

$$\upsilon_i^I(z_i)=\begin{cases}0, & if \quad z_i \le z_i^L,\\ \dfrac{z_i^U - z_i}{z_i^U - z_i^L}, & if \quad z_i^L \le z_i \le z_i^U.\\ 1, & if \quad z_i \ge z_i^U\end{cases} \tag{31}$$

4 Algorithm for Neutrosophic Multi-Objective Programming Problem

The computational procedure and proposed algorithm of presented model is given as follows:

Step 1. Determine $x_i^* = \left(x_{i1}^*, x_{i2}^*, ..., x_{in}^*\right)$ that is used to maximize or minimize the i^{th} truth membership function $\mu_i^I(X)$, the indeterminacy membership $\sigma_i^I(X)$, and the falsity membership functions $\upsilon_i^I(X)$. *i=1,2,..,p* and *n* is the number of variables.

Step 2. Transform the truth membership, indeterminacy membership, falsity membership functions by using first-order Taylor polynomial series

$$\mu_i^I(x) \cong \mu_i^I(x_i^*) + \sum_{j=1}^{n}\left(x_j - x_{ij}^*\right)\frac{\partial \mu_i^I(x_i^*)}{\partial x_j} \tag{32}$$

$$\sigma_i^I(x) \cong \sigma_i^I(x_i^*) + \sum_{j=1}^{n}\left(x_j - x_{ij}^*\right)\frac{\partial \sigma_i^I(x_i^*)}{\partial x_j} \tag{33}$$

$$\upsilon_i^I(x) \cong \upsilon_i^I\left(x_i^*\right) + \sum_{j=1}^{n}\left(x_j - x_{ij}^*\right)\frac{\partial \upsilon_i^I\left(x_i^*\right)}{\partial x_j} \tag{34}$$

Step 3. Find satisfactory $x_i^* = \left(x_{i1}^*, x_{i2}^*, ..., x_{in}^*\right)$ by solving the reduced problem to a single objective for the truth membership, indeterminacy membership, falsity membership functions respectively.

$$\begin{aligned}
p(x) &= \sum_{i=1}^{p}\left[\mu_i^I\left(x_i^*\right) + \sum_{j=1}^{n}\left(x_j - x_{ij}^*\right)\frac{\partial \mu_i^I\left(x_i^*\right)}{\partial x_j}\right] \\
q(x) &= \sum_{i=1}^{p}\left[\sigma_i^I\left(x_i^*\right) + \sum_{j=1}^{n}\left(x_j - x_{ij}^*\right)\frac{\partial \sigma_i^I\left(x_i^*\right)}{\partial x_j}\right] \\
h(x) &= \sum_{i=1}^{p}\left[\upsilon_i^I\left(x_i^*\right) + \sum_{j=1}^{n}\left(x_j - x_{ij}^*\right)\frac{\partial \upsilon_i^I\left(x_i^*\right)}{\partial x_j}\right]
\end{aligned} \tag{35}$$

Thus neutrosophic multiobjective linear programming problem is converted into a new mathematical model and is given below:

Maximize or Minimize *p(x)*

Maximize or Minimize *q(x)*

Maximize or Minimize *h(x)*,

where $\mu_i^I(X)$, $\sigma_i^I(X)$ and $\upsilon_i^I(X)$ calculate using equations (10), (11), and (12) or equations (13), (14), and (15) according to type functions maximum or minimum respectively.

4.1 Illustrative Example

A multi-criteria supplier selection is selected from [2]. For supplying a new product to a market assume that three suppliers should be managed. The purchasing criteria are net price, quality and service. The capacity constraints of suppliers are also considered.

It is assumed that the input data from suppliers' performance on these criteria are not known precisely. The neutrosophic values of their cost, quality and service level are presented in Table 1.

The multi-objective linear formulation of numerical example is presented as min z_1, max z_2, z_3:

$$\min\ z_1 = 5x_1 + 7x_2 + 4x_3,$$
$$\max\ z_2 = 0.80x_1 + 0.90x_2 + 0.85x_3,$$
$$\max\ z_3 = 0.90x_1 + 0.80x_2 + 0.85x_3,$$
$$s.t.:$$
$$x_1 + x_2 + x_3 = 800,$$
$$x_1 \le 400,$$
$$x_2 \le 450,$$
$$x_3 \le 450,$$
$$x_i \ge 0,\ i = 1,2,3.$$

Table 1: Suppliers quantitative information

	Z1 Cost	Z2Quality (%)	Z3 Service (%)	Capacity
Supplier 1	5	0.80	0.90	400
Supplier 2	7	0.90	0.80	450
Supplier 3	4	0.85	0.85	450

The truth membership, indeterminacy membership, falsity membership functions were considered to be neutrosophic triangular. When they depend on three scalar parameters (a1, m, a2). z_1 depends on neutrosophic aspiration levels (3550,4225,4900), when z_2 depends on neutrosophic aspiration levels (660,681.5,702.5), and z3 depends on neutrosophic aspiration levels (657.5,678.75,700).

The truth membership functions of the goals are obtained as follows:

$$\mu_1^I(z_1) = \begin{cases} 0, & if \quad z_1 \le 3550, \\ \dfrac{4225 - z_1}{4225 - 3550}, & if \quad 3550 \le z_1 \le 4225, \\ \dfrac{4900 - z_1}{4900 - 4225}, & if \quad 4225 \le z_1 \le 4900, \\ 0, & if \quad z_1 \ge 4900 \end{cases}$$

$$\mu_2^I(z_2) = \begin{cases} 0, & if \quad z_2 \ge 702.5, \\ \dfrac{z_2 - 681.5}{702.5 - 681.5}, & if \quad 681.5 \le z_2 \le 702.5, \\ \dfrac{z_2 - 660}{681.5 - 660}, & if \quad 660 \le z_2 \le 681.5, \\ 0, & if \quad z_2 \le 660. \end{cases}$$

$$\mu_3^I(z_3)=\begin{cases}0, & if \quad z_3 \geq 700,\\ \dfrac{z_3-678.75}{700-678.75}, & if \quad 678.75 \leq z_3 \leq 700,\\ \dfrac{z_3-657.5}{678.75-657.5}, & if \quad 657.5 \leq z_3 \leq 678.75,\\ 0, & if \quad z_3 \leq 657.5.\end{cases}$$

If $\mu_1^I(z_1)=\max\left(\min\left(\dfrac{4225-(5x_1+7x_2+4x_3)}{675},\dfrac{4900-(5x_1+7x_2+4x_3)}{675},0\right)\right)$

$$\mu_2^I(z_2)=\min(\max(\frac{(0.8x_1+0.9x_2+0.85x_3)-681.5}{21},\frac{(0.8x_1+0.9x_2+0.85x_3)-660}{21},1))$$

$$\mu_3^I(z_3)=\min(\max(\frac{(0.9x_1+0.8x_2+0.85x_3)-678.75}{21.25},\frac{(0.9x_1+0.8x_2+0.85x_3)-657.5}{21.25},1))$$

Then $\mu_1^{I*}(350,0,450)$, $\mu_2^{I*}(0,450,350)$, $\mu_3^{I*}(400,0,400)$

The truth membership functions are transformed by using first-order Taylor polynomial series

$$\hat{\mu}_1^I(x)=\mu_1^I(350,0,450)+\left[(x_1-350)\frac{\partial\mu_1^I(350,0,450)}{\partial x_1}\right]+\left[(x_2-0)\frac{\partial\mu_1^I(350,0,450)}{\partial x_2}\right]+\left[(x_3-450)\frac{\partial\mu_1^I(350,0,450)}{\partial x_3}\right]$$

$$\hat{\mu}_1^I(x)=-0.00741x_1-0.0104x_2-0.00593x_3+5.2611$$

In the similar way, we get

$$\hat{\mu}_2^I(x)=0.0381x_1+0.0429x_2+0.0405x_3-33.405$$

$$\hat{\mu}_3^I(x)=0.042x_1+0.037x_2+0.0395x_3-32.512$$

The *p(x)* is

$$p(x)=\hat{\mu}_1^I(x)+\hat{\mu}_2^I(x)+\hat{\mu}_3^I(x)$$

$$P(x) = 0.07259x_1 + 0.0695x_2 + 0.0741x_3 - 60.6559$$

$$\begin{aligned}
&s.t.:\\
&x_1 + x_2 + x_3 = 800,\\
&x_1 \le 400,\\
&x_2 \le 450,\\
&x_3 \le 450,\\
&x_i \ge 0,\ i = 1,2,3.
\end{aligned}$$

The linear programming software LINGO 15.0 is used to solve this problem. The problem is solved and the optimal solution for the truth membership model is obtained is as follows: (x_1, x_2, x_3) = (350, 0,450) z_1=3550, z_2=662. 5, z_3=697. 5.

The truth membership values are $\mu_1 = 1,\ \mu_2 = 0.1163,\ \mu_3 = 0.894.$ The truth membership function values show that both goals z_1 , $z_3$and z_2 are satisfied with 100% , 11.63% and 89.4% respectively for the obtained solution which is $x1$=350; $x2$= 0, x3=450.

In the similar way, we get $\sigma_i^I(X)$, *q(x).* Consequently, we get the optimal solution for the indeterminacy membership model is obtained is as follows: (x_1,x_2,x_3)=(350,0,450) z_1=3550, z_2=662.5, z_3=697.5 and the indeterminacy membership values are $\mu_1 = 1,\ \mu_2 = 0.1163,\ \mu_3 = 0.894.$ The indeterminacy membership function values show that both goals z_1 , $z_3$and z_2 are satisfied with 100% , 11.63% and 89.4% respectively for the obtained solution which is $x1$=350; $x2$= 0, x3=450.

In the similar way, we get $\upsilon_i^I(X)$ *and h(x)* Consequently, we get the optimal solution for the falsity membership model is obtained is as follows: (x_1,x_2,x_3)=(350,0,450) z_1=3550, z_2=662.5, z_3=697.5 and the falsity membership values are $\mu_1 = 0,\ \mu_2 = 0.8837,\ \mu_3 = 0.106.$ The falsity membership function values show that both goals z_1 , $z_3$and z_2 are satisfied with 0% , 88.37% and 10.6% respectively for the obtained solution which is $x1$=350; $x2$= 0, x3=450.

5 Conclusions and Future Work

In this chapter, we have proposed a solution to Neutrosophic Multiobjective Programming Problem (NMOPP). The truth membership, indeterminacy membership, falsity membership functions associated with each objective of the problem are transformed by using the first order Taylor polynomial series. The neutrosophic multi-objective programming problem is

reduced to an equivalent multiobjective programming problem by the proposed method. The solution obtained from this method is very near to the solution of MOPP. Hence this method gives a more accurate solution as compared with other methods. Therefore, the complexity in solving NMOPP has reduced to easy computation. In the future studies, the proposed algorithm can be solved by metaheuristic algorithms.

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VI

Multi-objective Cylindrical Skin Plate Design Optimization based on Neutrosophic Optimization Technique

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Abstract

In this chapter, we develop a Neutrosophic Optimization (NSO) approach for optimizing the thickness and sag of skin plate of vertical lift gate with multi- objective subject to a specified constraint. In this optimum design formulation, the objective function is the thickness and sag of the skin plate of vertical lift gate; the design variables are the thickness and sag of skin plate of vertical lift gate; the constraint are the stress and deflection in member. A classical vertical lift gate optimization example is presented here in to demonstrate the efficiency of this technique. The test problem includes skin plate of vertical lift gate subjected to hydraulic load condition. This multi-objective structural optimization model is solved by fuzzy, intuitionistic fuzzy and neutrosophic multi-objective optimization technique. Numerical example is given to illustrate our NSO approach. The result shows that the NSO approach is very efficient in finding the best discovered optimal solutions.

Keywords

Neutrosophic Set, Single Valued Neutrosophic Set, Neutrosophic Optimization, Cylindrical Skin Plate Design.

1 Introduction

Structural optimization is an important notion in civil engineering. Traditionally structural optimization is a well-known concept and in many situations, it is treated as single objective form, where the objective is known the weight function. The extension of this is the optimization where one or more constraints are simultaneously satisfied next to the minimization of the weight function. This does not always hold good in real world problems where multiple and conflicting objectives frequently exist. In this consequence, a methodology known as multi-objective structural optimization (MOSO) is introduced. In structural engineering design problems, the input data and parameters are often fuzzy/imprecise with nonlinear characteristics that necessitate the development of fuzzy optimum structural design method. Fuzzy set (FS) theory has long been introduced to handle inexact and imprecise data by Zadeh [2]. Later on Bellman and Zadeh [4] used the fuzzy set theory to the decision making problem. The fuzzy set theory also found application in structural design. Several researchers like Wang et al. [8] first applied α-cut method to structural designs where the non-linear problems were solved with various design levels α, and then a sequence of solutions were obtained by setting different level-cut value of α. Rao [3] applied the same α-cut method to design a four–bar mechanism for function generating problem. Structural optimization with fuzzy parameters was developed by Yeh et al. [9] Xu [10] used two-phase method for fuzzy optimization of structures. Shih et al. [5] used level-cut approach of the first and second kind for structural design optimization problems with fuzzy resources. Shih et al [6] developed an alternative α-level-cuts method for optimum structural design with fuzzy resources. Dey et al. [11] used generalized fuzzy number in context of a structural design. Dey et al. [13] developed parameterized t-norm based fuzzy optimization method for optimum structural design. Also, Dey et.al [16] Optimized shape design of structural model with imprecise coefficient by parametric geometric programming. In such extension, Atanassov [1] introduced Intuitionistic fuzzy set (IFS) which is one of the generalizations of fuzzy set theory and is characterized by a membership function, a non-membership function and a hesitancy function. In fuzzy sets, the degree of acceptance is only considered but IFS is characterized by a membership function and a non-membership function so that the sum of both values is less than one. A transportation model was solved by Jana et al [15] using multi-objective intuitionistic fuzzy linear programming. Dey et al. [12] solved two bar truss nonlinear problem by using intuitionistic fuzzy optimization problem. Dey et al. [14] used intuitionistic fuzzy optimization technique for multi objective optimum structural design. Intuitionistic fuzzy sets consider both truth membership and falsity membership. Intuitionistic fuzzy sets can only handle incomplete information not the indeterminate information and

inconsistent information. In neutrosophic sets indeterminacy is quantified explicitly and truth membership, indeterminacy membership and falsity membership which are independent. Neutrosophic theory was introduced by Smarandache [7]. The motivation of the present study is to give computational algorithm for solving multi-objective structural problem by single valued neutrosophic optimization approach. Neutrosophic optimization technique is very rare in application to structural optimization. We also aim to study the impact of truth membership, indeterminacy membership and falsity membership function in such optimization process. The results are compared numerically both in fuzzy optimization technique, intuitionistic fuzzy optimization technique and neutrosophic optimization technique. From our numerical result, it is clear that neutrosophic optimization technique provides better results than fuzzy optimization and intuitionistic fuzzy optimization.

2 Multi-objective Structural Model

In the design problem of the structure i.e. lightest thickness of the structure and minimum sag that satisfies all stress and deflection constraints in members of the structure. In vertical lift gate structural system, the basic parameters (including allowable stress ,deflection etc.) are known and the optimization's target is that identify the optimal thickness and sag so that the structure is of the smallest total weight with minimum stress and deflection in a given load conditions .

The multi-objective structural model can be expressed as:

$$\begin{aligned} &Minimize\ G \\ &minimize\ S \\ &subject\ to\ \sigma \le [\sigma] \\ &\delta \le [\delta] \\ &G^{\min} \le G \le G^{\max} \\ &S^{\min} \le S \le S^{\max} \end{aligned} \tag{1}$$

where G and S are the design variables for the structural design, δ is the deflection of the vertical lift gate of skin plate due to hydraulic load. σ is the stress constraint and $[\sigma]$, $[\delta]$ are allowable stress of the vertical lift gate of skin plate under various conditions. $G^{\min}$ and $S^{\min}$, $G^{\max}$ and $S^{\max}$ are the lower and upper bounds of design variables respectively.

3 Mathematical Preliminaries

3.1 Fuzzy Set

Let X be a fixed set. A fuzzy set: A set of X is an object having the form $\tilde{A} = \{(x, T_A(x)) : x \in X\}$ where the function $T_A : X \to [0,1]$ defined the truth membership of the element $x \in X$ to the set A.

3.2. Intuitionistic Fuzzy Set

Let a set X be fixed. An intuitionistic fuzzy set or IFS $\tilde{A}^i$ in X is an object of the form $\tilde{A}^i = \{< X, T_A(x), F_A(x) > | x \in X\}$ where $T_A : X \to [0,1]$ and $F_A : X \to [0,1]$ define the truth membership and falsity membership respectively, for every element of $x \in X$, $0 \le T_A(x) + F_A(x) \le 1$.

3.3. Neutrosophic Set

Let a set X be a space of points (objects) and $x \in X$ .A neutrosophic set $\tilde{A}^n$ in X is defined by a truth membership function $T_A(x)$, an indeterminacy-membership function $I_A(x)$ and a falsity membership function $F_A(x)$ and having the form $\tilde{A}^n = \{< x, T_A(x), I_A(x), F_A(x) > | x \in X\}$. $T_A(x), I_A(x)$ and $F_A(x)$ are real standard or non-standard subsets of $]0^-, 1^+[$ .That is

$$T_A(x) : X \to]0^-, 1^+[$$
$$I_A(x) : X \to]0^-, 1^+[$$
$$F_A(x) : X \to]0^-, 1^+[$$

There is no restriction on the sum of $T_A(x)$, $I_A(x)$ and $F_A(x)$ so $0^- \le \sup T_A(x) + \sup I_A(x) + \sup F_A(x) \le 3^+$.[17-22]

3.4. Single Valued Neutrosophic Set

Let a set X be the universe of discourse. A single valued neutrosophic set $\tilde{A}^n$ over X is an object having the form $\tilde{A}^n = \{< x, T_A(x), I_A(x), F_A(x) > | x \in X\}$ where:
$T_A : X \to [0,1], I_A : X \to [0,1], F_A : X \to [0,1]$ with $0 \le T_A(x) + I_A(x) + F_A(x) \le 3$ for all $x \in X$.

3.5. Complement of Neutrosophic Set

Complement of a single valued neutrosophic set A is denoted by $c(A)$ and is defined by:

$$T_{c(A)}(x) = F_A(x)$$
$$I_{c(A)}(x) = 1 - F_A(x)$$
$$F_{c(A)}(x) = T_A(x).$$

3.6. Union of Neutrosophic Sets

The union of two single valued neutrosophic sets A and B is a single valued neutrosophic set C, written as $C = A \cup B$,whose truth membership, indeterminacy-membership and falsity-membership functions are given by

$$T_{c(A)}(x) = \max\left(T_A(x), T_B(x)\right)$$
$$I_{c(A)}(x) = \max\left(I_A(x), I_B(x)\right)$$
$$F_{c(A)}(x) = \min\left(F_A(x), F_B(x)\right) \ \textit{for all}\ x \in X$$

3.7. Intersection of Neutrosophic Sets

The intersection of two single valued neutrosophic sets A and B is a single valued neutrosophic set C , written as $C = A \cap B$,whose truth membership, indeterminacy-membership and falsity-membership functions are given by

$$T_{c(A)}(x) = \min\left(T_A(x), T_B(x)\right)$$
$$I_{c(A)}(x) = \min\left(I_A(x), I_B(x)\right)$$
$$F_{c(A)}(x) = \max\left(F_A(x), F_B(x)\right) \ \textit{for all}\ x \in X\ .$$

4 Mathematical Analysis

4.1. Neutrosophic Optimization Technique to Solve Minimization Type Multi-Objective Non-linear Programming Problem

A nonlinear multi-objective optimization of the problem is of the form

$$\textit{Minimize}\left\{f_1(x), f_2(x), ..., f_p(x)\right\} \qquad (2)$$

Now the decision set $\tilde{D}^n$, a conjunction of Neutrosophic objectives and constraints is defined $\tilde{D}^n = \left(\bigcap_{k=1}^{p} \tilde{G}_k^n\right) \bigcap \left(\bigcap_{j=1}^{q} \tilde{C}_j^n\right) = \left\{\left(x, T_{\tilde{D}^n}(x)\right) I_{\tilde{D}^n}(x), F_{\tilde{D}^n}(x)\right\}$

$$\textit{Here}\ \ T_{\tilde{D}^n}(x) = \min\begin{Bmatrix} T_{\tilde{G}_1^n}(x), T_{\tilde{G}_2^n}(x), T_{\tilde{G}_3^n}(x), ..., T_{\tilde{G}_p^n}(x); \\ T_{\tilde{C}_1^n}(x), T_{\tilde{C}_2^n}(x), T_{\tilde{C}_3^n}(x), ..., T_{\tilde{C}_q^n}(x) \end{Bmatrix} \ \textit{for all}\ x \in X$$

$$I_{\tilde{D}^n}(x) = \min \begin{Bmatrix} I_{\tilde{G}_1^n}(x), I_{\tilde{G}_2^n}(x), I_{\tilde{G}_3^n}(x), ..., I_{\tilde{G}_p^n}(x); \\ I_{\tilde{C}_1^n}(x), I_{\tilde{C}_2^n}(x), I_{\tilde{C}_3^n}(x), ..., I_{\tilde{C}_q^n}(x) \end{Bmatrix} \quad for\ all\ x \in X$$

$$F_{\tilde{D}^n}(x) = \min \begin{Bmatrix} F_{\tilde{G}_1^n}(x), F_{\tilde{G}_2^n}(x), F_{\tilde{G}_3^n}(x), ..., F_{\tilde{G}_p^n}(x); \\ F_{\tilde{C}_1^n}(x), F_{\tilde{C}_2^n}(x), F_{\tilde{C}_3^n}(x), ..., F_{\tilde{C}_q^n}(x) \end{Bmatrix} \quad for\ all\ x \in X$$

where $T_{\tilde{D}^n}(x), I_{\tilde{D}^n}(x), F_{\tilde{D}^n}(x)$ are truth-membership function, indeterminacy membership function, falsity membership function of neutrosophic decision set respectively .Now using the neutrosophic optimization, problem (2) is transformed to the non-linear programming problem as

$$\begin{aligned} &Max\ \alpha \\ &Max\ \gamma \\ &Min\ \beta \\ &such\ that\ T_{\tilde{G}_k^n}(x) \geq \alpha;\ T_{\tilde{C}_j^n}(x) \geq \alpha;\ I_{\tilde{G}_k^n}(x) \geq \gamma;\ I_{\tilde{C}_j^n}(x) \geq \gamma; \\ &F_{\tilde{G}_k^n}(x) \leq \beta;\ F_{\tilde{C}_j^n}(x) \leq \beta;\ \alpha + \beta + \gamma \leq 3;\ \alpha \geq \beta; \alpha \geq \gamma; \\ &\alpha, \beta, \gamma \in [0,1] \end{aligned} \qquad (3)$$

Now this non-linear programming problem (3) can be easily solved by an appropriate mathematical programming to give solution of multi-objective non-linear programming problem (1) by neutrosophic optimization approach.

4.1.1 Computational Algorithm

Step-1: Solve the MONLP problem (2) as a single objective non-linear problem p times for each problem by taking one of the objectives at a time and ignoring the others. These solutions are known as ideal solutions. Let x^k be the respective optimal solution for the k^{th} different objective and evaluate each objective value for all these k^{th} optimal solution.

Step-2: From the result of step-1, determine the corresponding values for every objective for each derived solution, pay-off matrix can be formulated as follows

$$\begin{bmatrix} f_1^*(x^1) & f_2(x^1) & ..... & f_p(x^1) \\ f_1(x^2) & f_2^*(x^2) & ..... & f_p(x^2) \\ ..... & ..... & ..... & ..... \\ f_1(x^p) & f_2(x^p) & ..... & f_p^*(x^p) \end{bmatrix}$$

Step-3: For each objective $f_k(x)$ find lower bound L_k^μ and the upper bound U_k^μ

$$U_k^T = \max\left\{ f_k(x^{r*}) \right\} \quad and$$

$$L_k^T = \min\left\{ f_k(x^{r*}) \right\} \quad where\ r = 1, 2, ..., k$$

for truth membership of objectives.

Step-4: We represent upper and lower bounds for indeterminacy and falsity membership of objectives as follows:

for $k = 1, 2, ...p$

$U_k^F = U_k^T$ *and* $L_k^F = L_k^T + t\left(U_k^T - L_k^T\right);$

$L_k^I = L_k^T$ *and* $U_k^I = L_k^T + s\left(U_k^T - L_k^T\right)$

Here t, s are predetermined real numbers in $(0,1)$

Step-5: Define truth membership, indeterminacy membership and falsity membership functions as follows:

for $k = 1, 2, ..., p$

$$T_k\left(f_k(x)\right) = \begin{cases} 1 & \text{if } f_k(x) \le L_k^T \\ \dfrac{U_k^T - f_k(x)}{U_k^T - L_k^T} & \text{if } L_k^T \le f_k(x) \le U_k^T \\ 0 & \text{if } f_k(x) \ge U_k^T \end{cases}$$

$$I_k\left(f_k(x)\right) = \begin{cases} 1 & \text{if } f_k(x) \le L_k^I \\ \dfrac{U_k^I - f_k(x)}{U_k^I - L_k^I} & \text{if } L_k^I \le f_k(x) \le U_k^I \\ 0 & \text{if } f_k(x) \ge U_k^I \end{cases}$$

$$F_k\left(f_k(x)\right) = \begin{cases} 1 & \text{if } f_k(x) \le L_k^F \\ \dfrac{U_k^F - f_k(x)}{U_k^F - L_k^F} & \text{if } L_k^F \le f_k(x) \le U_k^F \\ 0 & \text{if } f_k(x) \ge U_k^F \end{cases}$$

Step-6: Now neutrosophic optimization method for MONLP problem gives a equivalent nonlinear programming problem as:

$$\text{Maximize } (\alpha - \beta + \gamma) \tag{4}$$

$$\text{such that } T_k\left(f_k(x)\right) \ge \alpha;\ I_k\left(f_k(x)\right) \ge \gamma;\ F_k\left(f_k(x)\right) \le \beta;$$

$$\alpha + \beta + \gamma \le 3;\ \alpha \ge \beta; \alpha \ge \gamma;\ \alpha, \beta, \gamma \in [0,1];$$

$$g_j(x) \le b_j \quad x \ge 0,\ k = 1, 2, ..., p;\ j = 1, 2, ..., q$$

This is reduced to equivalent nonlinear programming problem as

$$\text{Maximize } (\alpha - \beta + \gamma)$$

$$\text{such that } f_k(x) + \left(U_k^T - L_k^T\right).\alpha \le U_k^T;$$

$$f_k(x) + \left(U_k^I - L_k^I\right).\gamma \le U_k^I;$$

$$f_k(x) + \left(U_k^F - L_k^F\right).\beta \le L_k^F;$$

$$\text{for } k = 1, 2, ..., p$$

$$\alpha + \beta + \gamma \le 3;$$

$$\alpha \ge \beta; \alpha \ge \gamma;\ \alpha, \beta, \gamma \in [0,1];$$

$$g_j(x) \le b_j \quad x \ge 0. \tag{5}$$

5 Solution of Multi-Objective Structural Optimization Problem (MOSOP) by Neutrosophic Optimization Technique

To solve the MOSOP (1), step 1 of 4.1.1 is used. After that according to step to pay off matrix is formulated.

$$\begin{array}{c} \quad G \quad S \\ \begin{matrix} G^1 \\ S^2 \end{matrix}\begin{bmatrix} G^1 & S^1 \\ T^2 & S^2 \end{bmatrix} \end{array}$$

According to step-2 the bound of weight objective $U_G^T, L_G^T;\ U_G^I, L_G^I$ and U_G^F, L_G^F for truth, indeterminacy and falsity membership function respectively. Then:

$L_G^T \le G \le U_G^T;\ L_G^F \le G \le U_G^F;\ L_G^I \le G \le U_G^I$. Similarly, the bound of deflection objective are $U_S^T, L_S^T;\ U_S^F, L_S^F$ *and* U_S^I, L_S^I are respectively for truth, indeterminacy and falsity membership function.

Then:

$L_S^T \le S \le U_S^T;\ L_S^F \le S \le U_S^F;\ L_S^I \le S \le U_S^I$,

where $U_G^F = U_G^T;\ L_G^F = L_G^T + \varepsilon_G;\ L_G^I = L_G^T, U_G^I = L_G^T + \varepsilon_G$

and $U_S^F = U_S^T;\ L_S^F = L_S^T + \xi_S;\ L_S^I = L_S^T;\ U_S^I = L_S^T + \xi_S$,

such that

$0 < \varepsilon_G < \left(U_G^T - L_G^T\right)$ and $0 < \xi_S < \left(U_S^T - L_S^T\right)$.

According to neutrosophic optimization technique considering truth, indeterminacy and falsity membership function for MOSOP (1), and crisp non-linear programming problem can be formulated as

$$Maximize\ (\alpha + \gamma - \beta) \tag{6}$$

Subject to

$T_G \ge \alpha;\ T_S \ge \alpha;\ F_G \le \beta;\ F_G \le \beta;$

$I_G \ge \gamma;\ I_S \ge \gamma;\ \sigma \le [\sigma];\ \delta \le [\delta];$

$\alpha + \beta + \gamma \le 3;\ \alpha \ge \beta;\ \alpha \ge \gamma;$

$\alpha, \beta, \gamma \in [0,1],\quad G^{\min} \le G \le G^{\max}\ S^{\min} \le S \le S^{\max}$

Solving the above crisp model (6) by an appropriate mathematical programming algorithm we get optimal solution and hence objective functions i.e structural weight and deflection of the loaded joint will attain Pareto optimal solution.

6 Numerical Illustration

A cylindrical skin plate of vertical lift gate (Guha A.L et al [17]) in fig-2 has been considered. The weight of the skin plate is about 40% of the weight of the vertical lift gate, thus the minimum weight of the vertical lift gate can be achieved by using minimum thickness of a skin plate with same number of horizontal girders for the particular hydraulic load. It is proposed to replace stiffened flat skin plate by unstiffened cylindrical skin plate. The stress developed in skin plate and its distribution mainly depends on water head, skin plate thickness, and sag and position of Horizontal girders. Stress and deflection are expressed in terms of water head, skin plate thickness, and sag based on finite element analysis.

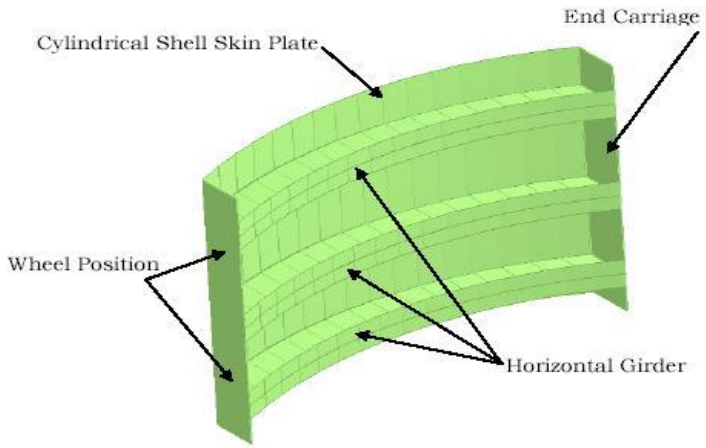

Fig. 1: Vertical lift gate with cylindrical shell type skin plate

The proposed expressions are furnished as stress $\sigma(G,S,H) = K_1 G^{-n_1} S^{-n_2} H^{n_3}$ where, $\sigma =$ stress in Kg/cm^2; $H =$ water Head in 'm' $G =$ Thickness in 'mm' $S =$ Sag in 'mm' $K_1 =$ Constant of variation and n_1; n_2 and $n_3 =$ constants depend on the properties of material Similarly, deflection: $\delta(T,S,H) = K_2 T^{-n_4} S^{-n_5} H^{n_6}$ $\delta(G,S,H) = K_2 G^{-n_4} S^{-n_5} H^{n_6}$ where, $K_2 =$ constant of variation and n_4; n_5 and $n_6 =$ constants depend on the properties of material.

To minimize the weight of Vertical gate by simultaneous minimization of Thickness G and sag, S of skin plate subject to maximum allowable stress (σ_0) and deflection (δ_0).

So, the model is

$$\text{Minimize } G \tag{7}$$

$$\text{Minimize } S$$

Subject to

$$\sigma(G,S,H) \equiv K_1 G^{-n_1} S^{-n_2} H^{n_3} \le \sigma_0;$$

$$\delta(G,S,H) \equiv K_2 G^{-n_4} S^{-n_5} H^{n_6} \le \delta_0$$

$$G, S > 0;$$

Input data of the problem is tabulated in Table. 1.

Table 1: Input data for crisp model (7)

constant of variation K_1	constant of variation K_2	constants depend on the properties of material	water head H (m)	Maximum allowable stress σ_0 (Mpa)	Maximum allowable deflection of girder δ_0 (Mpa)
3.79×10^{-3}	87.6×10^{-5}	$n_1 = 0.44;\ n_2 = 1.58;$ $n_3 = 1.0$ $n_4 = 0.729;\ n_5 = 0.895;$ $n_6 = 1.0$	25	137.5	5.5

Solution: According to step 2 of 4.1.1, pay-off matrix is formulated as follows:

$$\begin{array}{c} \\ G^1 \\ S^2 \end{array} \begin{array}{c} \begin{array}{cc} G & S \end{array} \\ \begin{bmatrix} 0.59\times10^{-5} & 37.61824 \\ 3528.536 & 0.10256\times10^{-2} \end{bmatrix} \end{array}$$

Here,

$U_G^F = U_G^T = 3528.536,\ L_G^F = L_G^T + \varepsilon_G = 0.59\times10^{-5} + \varepsilon_G;\ L_G^I = L_G^T = 0.59\times10^{-5},$
$U_G^I = L_G^T + \xi_G = 0.59\times10^{-5} + \xi_G$

such that $0 < \varepsilon_G, \xi_G < \left(3528.536 - 0.59\times10^{-5}\right)$;

$$U_S^F = U_S^T = 37.61824,\ L_S^F = L_S^T + \varepsilon_S = 0.10256\times10^{-2} + \varepsilon_S;$$
$$L_S^I = L_S^T = 0.10256\times10^{-2},\ U_S^I = L_S^T + \xi_S = 0.10256\times10^{-2} + \xi_S$$

such that $0 < \varepsilon_S, \xi_S < \left(37.61824 - 0.10256\times10^{-2}\right)$

Here, truth, indeterminacy, and falsity membership function for objective functions are G and S are defined as follows

$$T_G = \begin{cases} 1 & if\ \ G \le 0.59\times10^{-5} \\ \dfrac{3528.536 - G}{3528.536 - 0.59\times10^{-5}} & if\ \ 0.59\times10^{-5} \le G \le 3528.536 \\ 0 & if\ \ G \ge 3528.536 \end{cases}$$

$$F_G = \begin{cases} 0 & if\ \ G \le 0.59\times10^{-5} + \varepsilon_G \\ \dfrac{G - \left(0.59\times10^{-5} + \varepsilon_G\right)}{3528.536 - 0.59\times10^{-5} - \varepsilon_G} & if\ \ 0.59\times10^{-5} + \varepsilon_G \le G \le 3528.536; \\ 1 & if\ \ G \ge 3528.536 \end{cases}$$

$$I_G = \begin{cases} 1 & if\ \ G \le 0.59\times10^{-5} \\ \dfrac{\left(0.59\times10^{-5} + \xi_G\right) - G}{\xi_G} & if\ \ 0.59\times10^{-5} \le G \le 0.59\times10^{-5} + \xi_G \\ 0 & if\ \ G \ge 0.59\times10^{-5} + \xi_G \end{cases}$$

$$T_S = \begin{cases} 1 & if\ \ S \le 0.10256\times10^{-2} \\ \dfrac{37.61824 - S}{37.61824 - 0.10256\times10^{-2}} & if\ \ 0.10256\times10^{-2} \le S \le 37.61824; \\ 0 & if\ \ S \ge 37.61824 \end{cases}$$

$$F_S = \begin{cases} 0 & if\ \ S \le 0.10256\times10^{-2} + \varepsilon_S \\ \dfrac{S - \left(0.10256\times10^{-2} + \varepsilon_S\right)}{37.61824 - 0.10256\times10^{-2} - \varepsilon_S} & if\ \ 0.10256\times10^{-2} + \varepsilon_S \le S \le 37.61824; \\ 1 & if\ \ S \ge 37.61824 \end{cases}$$

$$I_S = \begin{cases} 1 & if\ \ S \le 0.10256\times10^{-2} \\ \dfrac{\left(0.10256\times10^{-2} + \xi_S\right) - S}{\xi_S} & if\ \ 0.10256\times10^{-2} \le S \le 0.10256\times10^{-2} + \xi_S \\ 0 & if\ \ S \ge 0.10256\times10^{-2} + \xi_S \end{cases}$$

Now using neutrosophic optimization technique with truth, indeterminacy and falsity membership functions we get

$$Maximize \quad (\alpha+\gamma-\beta)$$

$$subject\ to\ G+\left(3528.536-0.59\times10^{-5}\right)\alpha\le 3528.536;$$

$$S+\left(37.61824-0.10256\times10^{-2}\right)\alpha\le 37.61824;$$

$$G-(1-\beta)\left(0.59\times10^{-5}+\varepsilon_G\right)\le 3528.536\beta;$$

$$S-(1-\beta)\left(0.10256\times10^{-2}+\varepsilon_S\right)\le 37.61824\beta;$$

$$G+\xi_G\gamma\le\left(0.59\times10^{-5}+\xi_G\right);$$

$$S+\xi_S\gamma\le\left(0.10256\times10^{-2}+\xi_S\right);$$

$$\left(3.79\times10^{-3}\times25\right)G^{-0.44}S^{-1.58}\le 137.5;$$

$$\left(87.6\times10^{-5}\times25\right)G^{-0.729}S^{-0.895}\le 5.5;$$

$$\alpha\ge\beta;\ \alpha\ge\gamma;\ \alpha+\beta+\gamma\le 3;\ \alpha,\beta,\gamma\in[0,1]$$

Table 2: Comparison of Optimal solution of MOSOP (7) based on different method

Methods	G (mm)	T (mm)
Fuzzy multi-objective nonlinear programming (FMONLP)	52.88329	0.5648067
Intuitionistic fuzzy multi-objective nonlinear programming (IFMONLP) $\varepsilon_G=1764.268, \varepsilon_S=2.57033$	52.88329	0.5648065
Neutrosophic optimization (NSO) $\varepsilon_G=\xi_G=1764.268, \varepsilon_S=\xi_S=22.57033$	44.28802	0.5676034

Here we get best solutions for the different tolerance ξ_G, ξ_S for indeterminacy membership function of objective functions. From the table 2, it shows that NSO technique gives better Pareto optimal result in the perspective of Structural Optimization.

7 Conclusions

The main objective of this work is to illustrate how much neutrosophic optimization technique reduces thickness and sag of nonlinear vertical lift gate in comparison of fuzzy and intuitionistic fuzzy optimization technique. The concept of neutrosophic optimization technique allows one to define a degree of truth membership, which is not a complement of degree of falsity; rather, they are independent with degree of indeterminacy. Here we have considered a non-linear skin plate of vertical lift gate problem. In this problem, we find out minimum thickness of the structure as well as minimum sag of cylindrical skin plate. The comparisons of results obtained for the undertaken problem clearly show the superiority of neutrosophic optimization over fuzzy optimization and intuitionistic fuzzy optimization. The results of this study may lead to the development of effective neutrosophic technique for solving other model of nonlinear programming problem in different fields.

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VII

Neutrosophic MULTIMOORA: A Solution for the Standard Error in Information Sampling

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Abstract

If complete Data Mining is not possible one has to be satisfied with an information sample, as much representative as possible. The Belgian company "CIM" is doing marketing research for all Belgian newspapers, magazines and cinema. For some local newspapers, it arrives at a standard error of more than 15% or a spread of more than 30%, which is scientific nonsense but accepted by the publishers of advertisement. On the other side technical problems will ask for a much smaller standard deviation like for instance a standard error of 0.1% for the possibility that a dike is not strong enough for an eventual spring tide. Somewhat in between the usual standard error for marketing research is 5%. Is it possible to avoid this Spread by Sampling? Here Multi-Objective Optimization Methods may help. The Neutrosophic MULTIMOORA method, chosen for its robustness compared to many other competing methods, will solve the problems of normalization and of importance, whereas Fuzzy MULTIMOORA may take care of the annoying spread in the marketing samples. While an application on the construction of dwellings is given, many other applications remain possible like for Gallup polls concerning public opinion, general elections in particular.

Keywords

Neutrosophic MULTIMOORA, objectives, criteria, attributes, alternative solutions, decision matrix, weights, MOORA, Ratio System, Reference Point Method, Full Multiplicative Form, ordinal dominance, sample, standard error, spread, market research.

1 Introduction

Several solutions face different criteria expressed in different units, whereas the best outcome has to be found. Consider the following example of buying a new car. This car has to fulfill the following criteria:

1. The criterion "comfort" possesses the following attributes: excellent, medium, weak, for instance translated into the cardinal numbers: 2 for weak, 3 for medium and 4 for excellent, excellent being the double of weak (the translation of nominal words into cardinal numbers is very often exaggerated, see therefore e.g. Brauers et al. 2011).

2. The criterion "price" is expressed in $

3. The criterion "speed" is expressed in miles per hour

4. The criterion "shape" possesses the following attributes: ordinary and special, for instance translated into the cardinal numbers: 1 for ordinary and 2 for special.

In this example, the decision is made by one person. If the decision is rather coming from multi-persons it could be difficult to question the whole population concerned and one has to be satisfied by a sample representing the opinion of a group originated from face-to-face interviews till digital information. The distance between the opinion of the whole population and the sample is measured by the standard deviation in one direction and by the spread, being the double of the standard deviation, in both directions. If the publicity power of a newspaper, magazine, cinema or television would be announced by these media themselves the public, especially the publicity brokers, would have no confidence in the outcome. Therefore, a neutral institution will deliver the results by sampling. CIM is for instance the organization concerned in Belgium.

The Association for measuring the importance of Newspapers in Belgium (CIM) is only interested in the evolution of the sales of newspapers in Belgium. It means that not the size of sales is of importance but rather its evolution. Indeed, beside the national papers with high volume local or specialized newspapers are not aiming at coming on the volume of the national papers but are interested in the increase or decrease of the number of readers. Nevertheless, one has to go out

from the numbers of readers to deduct the evolution in reader's population. The fact that a higher standard deviation is noticed for the local or specialized newspapers is another remarkable fact to be taken into consideration. Results for 2013-14 are synthesized in the following table with an average spread for all newspapers together of 24% (CIM September 2014 and CIM 2013-14).

Table 1. Average readership of Belgian newspapers (2013-14)

Newspapers	Circulation	Standard deviation	Spread
National newspaper (Dutch)	250,000	5.4%	10.8%
National newspaper (French)	180,000	6%	12%
Local newspaper (French)	40,000	15%	30%
Financial newspaper	46,000	6%	12%

The results are scientifically not acceptable but the publicity brokers prefer these results above eventual statistics from the newspapers themselves.

The topic of this research is to find a method in such a multi criteria problem of sampling in order to make a choice in a rational way, to come to an optimum for the results and to interpret them. More specific it concerns here market research.

In summary, one may say that first a method is needed to compare several criteria expressed in different units, secondly how to make a sample representative and thirdly how to deal with group decisions. First a method to compare the different criteria is searched out.

2 Search for a Robust Method to Make a Choice in a Rational Way between Different Solutions Responding to Different Objectives

For the researcher in multi-objective decision making the choice between many methods is not very easy. Indeed numerous theories were developed since the forerunners: Condorcet (the Condorcet Paradox, against binary comparisons, 1785, LVIII), Gossen (law of decreasing marginal utility, 1853) Minkowski (Reference Point, 1896, 1911) and Pareto (Pareto Optimum and Indifference Curves analysis 1906, 1927) and pioneers like Kendall (ordinal scales, since 1948), Roy (ELECTRE, since 1966, with many variations in Electre since then, see therefore Schärlig, 1985; 1996), Miller and Starr (Multiplicative Form, 1969), Hwang and Yoon (TOPSIS, 1981), Saaty (AHP, since 1988), Opricovic and Tzeng (VIKOR, 2004), Brans and Mareschal (PROMETHEE, 2005).

The MULTIMOORA (Multi-Objective Optimization by a Ratio Analysis plus the Full Multiplicative Form) was proposed by Brauers and Zavadskas (2010).

The ordinary MULTIMOORA method has been proposed for usage with crisp numbers. To enable its use in solving a larger number of complex decision-making problems, several extensions have been proposed, from which there are mentioned only the most prominent: Brauers *et al.* (2011) proposed fuzzy extension of the MULTIMOORA method; Balezentis and Zeng (2013) proposed interval-valued fuzzy, Balezentis *et al*. (2014) proposed intuitionist fuzzy extension and Zavadskas *et al.* (2015) proposed interval-valued intuitionist extension of the MULTIMOORA method.

A significant approach in solving complex decision-making problems was formed by adapting multiple criteria decision-making methods for the use of fuzzy numbers, proposed by Zadeh in fuzzy set theory (Zadeh, 1965).

Based on fuzzy set theory, some extensions are also proposed, such as: interval-valued fuzzy sets (Turksen, 1986), intuitionist fuzzy sets (Atanassov, 1986) and interval-valued intuitionist fuzzy sets (Atanassov, Gargov, 1989).

In addition to membership function, proposed in fuzzy sets, Atanassov (1986) introduced the non-membership function that express the non-membership to a set, and thus created the basis for solving of a much larger number of decision-making problems.

The intuitionist fuzzy set is composed of the membership (or called truth-membership) $T_A(x)$ and non-membership (or called falsity-membership) $F_A(x)$, that satisfies the conditions $T_A(x), F_A(x) \in [0,1]$ and $0 \le T_A(x) + F_A(x) \le 1$. Therefore, intuitionist fuzzy sets are capable to operate with incomplete information, but do not include intermediate and inconsistent information (Li *et al*., 2016).

In intuitionist fuzzy sets, the indeterminacy $\pi_A(x)$ is $1 - T_A(x) - F_A(x)$ by default. Smarandache (1999) further extended intuitionist fuzzy sets by proposing Neutrosophic, and also introduce independent indeterminacy-membership.

Such proposed neutrosophic set is composed of three independent membership functions named the truth-membership $T_A(x)$, falsity-membership $F_A(x)$ and indeterminacy-membership $I_A(x)$. *(Mohamed et al. , 2014,2015,2016a, 2016b,2017).*

Wang *et al*. (2010) further proposed a single valued neutrosophic set, by modifying the condition $T_A(x)$, $I_A(x)$ and $F_A(x) \in [0,1]$ and $0 \le T_A(x) +$

$I_A(x)+F_A(x)\leq 3$, which are more suitable for solving scientific and engineering problems (Li *et al*., 2016).

Compared with the fuzzy set and its extensions, the single valued neutrosophic set can be identified as more flexible, for which reason an extension of the MULTIMORA method adapted for the use of single valued neutrosophic set is proposed in this approach.

3 The Neutrosophic Extension of MULTIMOORA

A Decision Matrix assembles raw data with vertically numerous objectives, criteria (a weaker form of objectives) or indicators and horizontally alternative solutions, like projects. In order to define an objective better we have to focus on the notion of Attribute. Keeney and Raiffa (1993, 32-38) present the example of the objective "reduce sulfur dioxide emissions" to be measured by the attribute "tons of sulfur dioxide emitted per year". An attribute is a common characteristic of each alternative such as its economic, social, cultural or ecological significance, whereas an objective consists in the optimization (maximization or minimization) of an attribute.

3.1. Horizontal reading of the Decision Matrix

SAW, followed by many other methods, reads the response matrix in a horizontal way. The Additive Weighting Procedure (MacCrimmon, 1968, 29-33, which was called SAW, Simple Additive Weighting Method, by Hwang and Yoon, 1981, 99) starts from:

$$Max.U_j = w_1 x_{1j} + w_2 x_{2j} + \ldots + w_i x_{ij} + \ldots + w_n x_{nj} \quad (1)$$

Uj = overall utility of alternative j with j = 1,2,…..,m, m the number of alternatives

wi = weight of attribute i indicates as well as normalization as the level of importance of an objective, with:

$$\sum_{i=1}^{i=n} w_i = 1$$

i = 1, 2,… n; n the number of attributes or objectives
xij = response of alternative j on attribute i.

As the weights add to one a new super-objective is created and consequently it becomes difficult to speak of multiple objectives.

With weights importance of objectives is mixed with normalization. Indeed, weights are mixtures of normalization of different units and of importance coefficients.

3.2. Vertical Reading of the Decision Matrix

Vertical reading of the Decision Matrix means that normalization is not needed as each column is expressed in the same unit. In addition, if each column is translated in ratios dimensionless measures can be created and the columns become comparable to each other. Indeed, they are no more expressed in a unit. Different kind of ratios are possible but Brauers and Zavadskas (2006) proved that the best one is based on the square root in the denominator.

Vertical reading of the decision matrix and the Brauers-Zavadskas ratios are practiced in the MOORA method.

3.3. The MOORA Method

3.3.1. Ratio System of MOORA

We go for a ratio system in which each response of an alternative on an objective is compared to a denominator, which is representative for all alternatives concerning that objective:

$$x_{ij}^{*} = \frac{x_{ij}}{\sqrt{\sum_{j=i}^{m} x_{ij}^{2}}} \tag{2}$$

with:

xij = response of alternative j on objective i
$j = 1,2,\ldots,m;$ m the number of alternatives
$i = 1,2,\ldots n;$ n the number of objectives
$xij*=$ this time a dimensionless number representing the response of alternative j on objective i. $xij*$ is situated between 0 and 1[1].

[1] However, sometimes the interval could be *[-1; 1]*. Indeed, for instance in the case of productivity growth some sectors, regions or countries may show a decrease instead of an increase in productivity i.e. a negative dimensionless number. Instead of a normal increase in productivity growth a decrease remains possible. At that moment, the interval becomes *[-1, 1]*. Take the example of productivity, which has to increase (positive). Consequently, we look for a maximization of productivity e.g. in European and American countries. What if the opposite does occur? For instance, take the original transition from the USSR to Russia. Contrary to the other European countries productivity decreased. It means that in formula (2) the numerator for Russia was negative with the whole ratio becoming negative. Consequently, the interval changes to: *[-1, +1]* instead of *[0, 1]*.

For optimization, these responses are added in case of maximization and subtracted in case of minimization:

$$y_j^* = \sum_{i=1}^{i=g} x_{ij}^* - \sum_{i=g+1}^{i=n} x_{ij}^* \quad (3)$$

with:

$i = 1,2,\dots,g$ as the objectives to be maximized.

$i = g+1, g+2, \dots, n$ as the objectives to be minimized

y_j* = the total assessment of alternative j with respect to all objectives.

y_j* can be positive or negative depending of the totals of its maxima and minima.

An ordinal ranking of the y_j* in a descending order shows the final preference. Indeed, cardinal scales can be compared in an ordinal ranking after Kenneth J. Arrow (1974): "Obviously, a cardinal utility implies an ordinal preference but not *vice versa"*.

A second part of MOORA consists of the Reference Point Method which uses the ratios found in the Ratio System of MOORA.

3.3.2. Reference Point method of MOORA

A second Method in MOORA is the Reference Point Approach which will use the ratios found earlier and whereby also a Maximal Objective Reference Point is used. The Maximal Objective Reference Point approach is called realistic and non-subjective as the co-ordinates (r_i), which are selected for the reference point, are realized in one of the candidate alternatives. In the example, A (10;100), B (100;20) and C (50;50), the maximal objective reference point R_m results in: (100;100). Per objective the coordinates of the corresponding ratio are subtracted from the coordinates of the Reference Point.

Then these results are subject to the *Metric of Tchebycheff* (Karlin and Studden, 1966, 280)[2]:

$$\min_{(j)} \left\{ \max_{(i)} \sqrt[2]{(r_i - x_{ij}^*)^2} \right\} \quad (4)$$

ri = the i^{th} co-ordinate of the reference point

$xij*$= the dimensionless measurement of objective i for alternative j

$i = 1,2.....n$; n the number of objectives

$j = 1,2.....m$; m the number of alternatives

[2] Previously the following formula was used as Tchebycheff has done:

$$\min_{(j)} \left\{ \max_{(i)} | \, r_i - x_{ij}^* \, | \right\}$$

with $| \, r_i - x_{ij}^* \, |$ the absolute value necessary if x_{ij}^* is larger than r_i

The outcome is the same, but the square presentation (4) is more in accordance with formula (2).

An ordinal ranking of the results in an ascending order shows the final preference.

3.3.3. The problem of importance

With weights importance of objectives is mixed with normalization. On the contrary the dimensionless measures of MOORA do not need external normalization. However, the problem of importance remains. Therefore, in MOORA to give more importance to an objective its response on an alternative under the form of a dimensionless number could be multiplied with a significance coefficient. However, if this would be done the outcome will not change. Therefore, another approach has to be followed. Replacement of an objective by some sub-objectives, as valuable as the original objectives, will solve the problem of importance for the original objective. For instance, employment is replaced separately by direct and indirect employment or pollution is divided into three different forms of pollution.

3.3.4. MOORA can it be called Robust? Characteristics of Robustness in Multi-Objective Optimization

(Brauers, 2010; Brauers and Zavadskas; 2012; Brauers and Zavadskas, 2010; Brauers and Zavadskas, 2009; Brauers and Ginevičius, 2010; Brauers and Ginevičius, 2009)

1. All stakeholders are involved (see: Brauers and Lepkova, 2003 and 2002).
2. Respect for Consumer Sovereignty (Brauers, 2008b)
3. All non-correlated objectives are involved, as much as possible (see Brauers et al. 2008)
4. All interrelations between objectives and alternatives are considered at the same time and for instance not two by two (otherwise a victim of the Condorcet-Arrow Paradox, see: Brauers, 2004, 118-124).
5. Non-subjective as much as possible:
   - In the choice of the objectives (assistance can be given by the Ameliorated Nominal Group Technique, see Brauers, 2008a; Brauers and Lepkova 2003 and 2002)
   - To give importance to an objective either in a direct way or by substitution (assistance can be given by the Delphi Method, see Brauers, 2008a; Brauers, 1976; Dalkey and Helmer, 1963)
   - Omitting Normalization. Dimensionless Measurements as used here are preferred to weights, which need normalization (for normalization, see: Brauers and Zavadskas 2007; Brauers 2007a and b).

6. Based on Cardinal Numbers is more robust than on Ordinal Numbers. The Rank Correlation Method of Kendall is based on ordinal numbers. He argues (Kendall, 1948, 1): "we shall often operate with these numbers as if they were the cardinals of ordinary arithmetic, adding them, subtracting them and even multiplying them", but he never gave a proof of this statement. In his later work this statement is dropped (Kendall and Gibbons, 1990).
7. Uses the most recent available data.
8. The use of two different methods of MOO is more robust than using a single one.

Already in 1983 at least 96 methods for Multi-Objective Optimization existed (Despontin et al., 1983). Since then numerous other methods appeared. Therefore, we only cite the probably most used methods for Multi-Objective Optimization.

First Schärlig (1985, 1996) gives the name of Methods of Partial Aggregation to the Electre Group (Electre I, Electre Iv, Electre Is, Electre TRI, Electre II, Electre III and Electre IV) and to Prométhée. As the study under consideration asks for total aggregation methods based on partial aggregation cannot be used.

The Analytic Hierarchy Process (AHP of Saaty, 1988), followed by the Analytic Network Process (ANP, Saaty & Kulakowski, 2016), compare in pairs and are based on weights. The use of weights in operational research was introduced by Churchman and Ackoff (1954) and Churchman et al. (1957). The Additive Weighting Procedure called SAW was already mentioned. Also, the methods of partial aggregation use weights. In addition, all these methods are expert oriented with qualitative statements as a basis.

Reference Point Methods like TOPSIS (Hwang and Yoon (1981) and VIKOR (Opricovic, Tzeng 2004) do not use weights but rather dimensionless measures but they are overtaken by MOORA which is composed of two different dimensionless based methods, each controlling each other.

An interesting example of MOORA compared with other methods is what Chakraborty has done for industrial management. Chakraborty (2011) checked six famous methods of Multi-Objective Decision Making for decision making in manufacturing. Next Table 1 shows the results.

Table 2. Comparative performance of some MODM methods

MODM	Computational time	Simplicity	Mathematical Calculations	Stability	Information Type
MOORA	Very less	Very simple Very critical	Minimum	Good	Quantitative
AHP	Very high	Moderately critical	Maximum	Poor	Mixed
TOPSIS	Moderate		Moderate	Medium	Quantitative
VIKOR	Less	Simple	Moderate	Medium	Quantitative
ELECTRE	High	Moderately critical	Moderate	Medium	Mixed
PROMETHEE	High	Moderately critical	Moderate	Medium	Mixed

Karuppanna & Sekar (2016, 61) studied the several approaches not only towards Manufacturing but also to the Service Sectors, which is extremely important for the underlying study.

Table 3. Comparison of MOORA with other Approaches for application in the Service Sectors

MADM method	Computational Time	Simplicity	Mathematical calculations
MOORA	very less	very simple	Minimum
AHP	very high	very critical	Maximum
ANP	Moderate	Moderately critical	Moderate
VIKOR	Less	Simple	Moderate
TOPSIS	Moderate	Moderately critical	Moderate
ELECTRE	High	Moderately critical	Moderate
PROMOTHEE	High	Moderately critical	Moderate

3.3.5. MOORA and Market Research

Market research works mostly with a confidence level of 95%, which means a 5% probability that outside conditions will interfere. On the other side for instance a dam against flooding has to have a confidence level of 99.9%, i.e. a probability of 1 on 1,000 that the dam will be too low or will collapse.

On the other side, the size of the sample is important. Marketing accepts for instance 100 interviews with a standard error of: se = $\sqrt{\frac{pq}{n}} = \sqrt{\frac{0.25}{100}} =$ 0.05 which means 5% under or 5% above the real percentage (p = expected probability; q the opposite q = 1 – p).

In a normal distribution: $q = p = 0.5$. The sum of the 5% under plus the 5% above the real percentage or the sum of the standard errors is called the Spread. Hoel (1971, 101) speaks of the extent of the spread, whereas Hays (1973, 236) calls it spread or dispersion. Mueller et al. (1970) speak rather of "Range".

3.3.6. Consumer's Attitude on Contractor's Ranking: a Presentation of a Case Study

This example is taken from: Brauers et al., 2008. Construction, taking off, maintenance and facilities management of a building are typical examples of consumer sovereignty: the new owner likes to have a reasonable price to pay, to have confidence in the contractor, to know about the duration of the works, the service after completion and the quality of the work. On the other side, the contractor has his objectives too, like the satisfaction of the client, diminishing of external costs and annoyances and the management cost per employee as low as possible. In other words, it concerns a problem of multi-objectives. Therefore, a final ranking will show the best performing contractor from the point of view of the clients but also from the point of view of the contractors.

The largest maintenance contractors of dwellings in Vilnius, the capital of Lithuania, were approached, of which 15 agreed to fix and estimate their main objectives, namely 9 objectives as given in Table 4.

Table 4**.** Main attributes and objectives of maintenance contractors of dwellings in Vilnius

1. Cost of building management Lt/m2 *min*
2. Cost of common assets management Lt/m2 *min*
3. HVAC system maintenance cost (mean) Lt/m2 *min*
4. Courtyard territory cleaning (in summer) Lt/m2 *min*
5. Total service cost Lt/m2 *min*
6. Length of time in maintenance business experience in years *max*
7. Market share for each contractor % *max*
8. Number of projects per executive units/person *max*
9. Evaluation of management cost (Cmin / Cp) *max*

Table 5 summarizes the reaction of the contractors on the proposed objectives.

Table 5**.** Initial decision making matrix of 15 contractors of dwellings in Vilnius

Objectives ↓ / Alternatives ↔	1	2	3	4	5	6	7	8	9
	MIN.	**MIN.**	**MIN.**	**MIN.**	**MIN.**	**MAX.**	**MAX.**	**MAX.**	**MAX.**
a_1	0.064	0.11	0.18	0.31	0.67	12	11.75	4.6	0.83
a_2	0.06	0.14	0.37	0.12	0.5	3	0.39	0.33	0.885
a_3	0.057	0.11	0.18	0.15	0.69	12	5.25	1.47	0.935
a_4	0.06	0.12	0.10	0.15	0.57	12	7.1	2.78	0.9
a_5	0.058	0.1	0.18	0.2	0.45	12	5.56	1.39	0.9
a_6	0.071	0.3	0.18	0.26	0.82	13	26.62	5.67	0.746
a_7	0.11	0.14	0.18	0.12	0.55	5	2.82	1.2	0.483
a_8	0.058	0.18	0.37	0.19	0.61	11	9.48	3.03	0.916
a_9	0.053	0.14	0.16	0.23	0.8	11	2.23	0.8	1
a_{10}	0.07	0.26	0.29	0.2	0.7	11	13.5	9.05	0.75
a_{11}	0.12	0.2	0.09	0.2	0.81	4	4.7	1.5	0.443
a_{12}	0.071	0.28	0.18	0.28	0.73	12	2.35	0.86	0.746
a_{13}	0.078	0.2	0.18	0.3	0.76	8	5.6	3.25	0.681
a_{14}	0.056	0.14	0.18	0.12	0.5	11	2.66	1.7	0.948
a_{15}	0.12	0.14	0.09	0.21	0.56	3	0.04	0.03	0.531

Brauers et al, 2008, 250.

From information of the Dwelling Owners Association, a panel of 30 owners of dwellings chosen at random agreed with these 9 objectives, but they increased the objectives with 11 other ones (These additional objectives were: standard of management services, maintenance of common property, work organization, effectiveness of information use, certification of company, range of services, reliability of company, company reputation, staff qualification and past experience, communication skills, geographical market restrictions.). However, these additional objectives were only expressed in qualitative points showing some overlapping and after their rating represented only 25.9% importance of the total. If these opinions are only taken as indicative these qualitative objectives can be dropped[3].

For the 9 objectives with 30 interviews even chosen at random mean a confidence level of: standard error se = $\sqrt{\frac{pq}{n}} = \sqrt{\frac{0.25}{30}} = 0.09$, which means 9% under or 9% above the real percentage or a Spread of 18%.

[3] Another approach would be that the corresponding ordinal qualifications are transformed into cardinal numbers, which has to be done under severe reservations. See therefore: Brauers, 2004, 97-99 and Brauers et al. 2011, 268-271.

Beside this formula: one has to be aware of the *Universe* or *Population* around the sample (Mueller et al. 1970, 343) which is not directly quantitative:

- Only the Vilnius population above the age of 18 has to be taken into consideration and in addition only households;
- an advance payment for buying property of 15 to 30% is needed in Lithuania (Swedbank, 2012);
- In addition, only 13% of the Vilnius population have a mortgage (SEB, 2013,6). From this 13% has to be excluded: existing mortgages, buying an existing property, buying a social apartment or people not interested in the location in question;
- Saving rate in Lithuania was only 1.92% in 2008, which is extremely low. In 2009 there was even dissaving (Statistics Lithuania, 2014).

Accepting the 18% spread for a limited universe one may conclude that the 30 respondents are representative for the potential buyers of the proposed property in Vilnius.

The nature of the construction industry involves that the total number of the minima is mostly larger than the total number of the maxima, which is the case here. Instead of attributing significance coefficients the contractors and the small sample of owners preferred the *Attribution of Sub-Objectives*. Indeed, five objectives on nine concern the super objective minimization of costs. Even, the last maximization forms in fact a cost consideration.

The following table 6 presents the ranking of the contractors.

Table 6. Ranking Contractors after MOORA with 18% spread (a)

	Ratio method	Reference Point Method	MOORA
a6	1	1	1
a10	2	2	2
a1	4	3	3
a4	3	5	4

Brauers et al, 2008, 251.

The other 10 contractors have a low rate and are unclear about their ranking.

A summary of the two methods in order to come to MOORA is made on view.

The problem remains of the high spread of 18%. How to solve the failure of:

- The high spread of 18%
- An unclear ranking?

Therefore, we look at MULTIMOORA and Fuzzy MULTIMOORA.

4 How to make a Sample representative without Spread?

4.1. The MULTIMOORA Method

To the two methods of MOORA a third method is added: the Full Multiplicative Form. The use of three different methods of MOO is more robust than using of two, making MULTIMOORA superior to all existing methods of Multiple Objectives Optimization.

In the Full Multiplicative Form per row of an alternative all objectives are simply multiplied, but the objectives to be minimized are parts of the multiplication process as denominators.

A problem may arise for a single zero or for a negative number for one of the objectives making the final product zero or entirely negative. In order to escape of this nonsense solution 0.001 replaces zero, if the lowest number present is 0.01. For a negative number, which will be very exceptional, see a case in footnote 1 above, -1 becomes 0.0001 and -2 becomes 0.00001 etc. but only for the objective under consideration.

In MOORA a summary of the two methods was made on view, impossible for MULTIMOORA. Adding of ranks, ranks mean an ordinal scale (1st, 2nd, 3rd etc.) signifies a return to a cardinal operation (1 + 2 +3 + …). Is this allowed? The answer is “no” following the Noble Prize Winner Arrow:

The Impossibility Theorem of Arrow

“Obviously, a cardinal utility implies an ordinal preference but not *vice versa”* (Arrow 1974).

Axioms on Ordinal and Cardinal Scales

1. A deduction of an Ordinal Scale, a ranking, from cardinal data is always possible.
2. An Ordinal Scale can never produce a series of cardinal numbers.
3. An Ordinal Scale of a certain kind, a ranking, can be translated in an ordinal scale of another kind.

In application of axiom 3 the rankings of three methods of MULTIMOORA are translated into another ordinal scale based on *Dominance, being Dominated, Transitivity and Equability*.

4.2. Ordinal Dominance Theory

4.2.1 Dominance

Absolute Dominance means that an alternative, solution or project is dominating in ranking all other alternatives, solutions or projects which are all being dominated. This absolute dominance shows as rankings for MULTIMOORA: (1– 1– 1). *General Dominance in two of the three methods* is of the form with a < b < c <d:

$(d-a-a)$ is generally dominating $(c-b-b)$;

$(a-d-a)$ is generally dominating $(b-c-b)$;

$(a-a-d)$ is generally dominating $(b-b-c)$;

And further transitiveness plays fully.

Transitiveness. If *a* dominates *b* and *b* dominates *c* than also *a* will dominate *c*.

Overall Dominance of one alternative on the next one. For instance $(a-a-a)$ is overall dominating $(b-b-b)$ which is overall being dominated.

1.2 2 Equability

Absolute Equability has the form**:** for instance (e–e–e) for 2 alternatives. *Partial Equability* of 2 on 3 exists e. g. (5–e–7) and (6–e–3).

4.3. MULTIMOORA with spread

Table 7. Ranking Contractors after MULTIMOORA with 18% spread (a)

Original study: MOORA with 18% spread		MULTIMOORA with 18% spread	
a6	1	a6	**1**
a10	2	a4	**2**
a1	3	a10	**3**
a4	4	a1	**4**
		a5	**5**
		a3	**6**
		a8	**7**
		a14	**8**
		a13	**9**
		a9	**10**
		a7	**11**
		a11	**12**
		a12	**13**
		a2	**14**
		a15	**15**

Calculations available from the authors

The spread still remains in MULTIMOORA. Fuzzy MULTIMOORA will try to remove the spread by extending the numbers until the standard deviation on both sides as given in next table 8.

Table 8. Ranking Contractors with 9% less and 9% more for each objective

	Obj. 1		Obj. 2.... Obj. 3 ...Obj. 4...Obj. 5....Obj. 6....Obj. 7....Obj. 8...	Obj. 9		
0.058	0.064	0.070	----------	0.76	0.83	0.90
0.055	0.060	0.065	----------	0.81	0.89	0.96
0.052	0.057	0.062	----------	0.85	0.94	1.02
0.053	0.058	0.063	----------	0.83	0.91	0.99
0.053	0.058	0.063	----------	0.83	0.91	0.99
0.065	0.071	0.077	----------	0.68	0.75	0.81
0.100	0.110	0.120	----------	0.44	0.48	0.53
0.053	0.058	0.063	----------	0.83	0.92	1.00
0.048	0.053	0.058	----------	0.91	1.00	1.09
0.065	0.071	0.077	----------	0.68	0.75	0.81
0.109	0.120	0.131	----------	0.40	0.44	0.48
0.065	0.071	0.077	----------	0.68	0.75	0.81
0.071	0.078	0.085	----------	0.62	0.68	0.74
0.051	0.056	0.061	----------	0.86	0.95	1.03
0.109	0.120	0.131	----------	0.48	0.53	0.58

27 objectives and sub-objectives replace the 9 objectives.

Consumer Sovereignty will play by giving to each objective a minus value or a max value of 9% deviation corresponding with the confidence level. For instance, input of contractor a1 into objective 6 being 12 is replaced by 10.92, 12 and 13.08 (see table 5).

In taking rows and columns in table 8 the numbers will have more or less the form of an upside-down Gauss Curve, however not standard normal or symmetrical (Hoel, 1971, 100-104) but skewed (Hays, 1973, 317) and with the restriction that the solutions are not continuous but discrete. Fuzzy means also that all points on a line linking all values of an alternative solution, here a contractor, are also possible.

In the given example, it is not certain that a contractor will accept the changes, proposed by the client, as it means a change in his offer.

4.4. The Fuzzy MULTIMOORA Method

Being a special case of the fuzzy sets, fuzzy numbers express uncertain quantities. Among various instances of fuzzy numbers, the triangular fuzzy numbers are often used for multi-criteria decision making. A triangular fuzzy

number $\tilde{x}$ can be represented by a tripet: $\tilde{x} = (a,b,c)$, where a and c are the minimum and maximum bounds, respectively, and b is the modal value or kernel (Kaufmann and Gupta, 1991).

The following arithmetic operations are available for the fuzzy numbers (Wang, Chang, 2007):

1. Addition $\oplus$:

$$\tilde{A} \oplus \tilde{B} = (a,b,c) \oplus (d,e,f) = (a+d, b+e, c+f); \quad (5)$$

2. Subtraction $\ominus$:

$$\tilde{A} \ominus \tilde{B} = (a,b,c) \ominus (d,e,f) = (a-f, b-e, c-d); \quad (6)$$

3. Multiplication $\otimes$:

$$\tilde{A} \otimes \tilde{B} = (a,b,c) \otimes (d,e,f) = (a \times d, b \times e, c \times f) \quad (7)$$

4. Division $\oslash$:

$$\tilde{A} \oslash \tilde{B} = (a,b,c) \oslash (d,e,f) = (a \setminus f, b \setminus e, c \setminus d). \quad (8)$$

The vertex method can be applied to measure the distance between two fuzzy numbers. Let $\tilde{A} = (a,b,c)$ and $\tilde{B} = (d,e,f)$ be the two triangular fuzzy numbers. Then, the vertex method can be applied:

$$d(\tilde{A}, \tilde{B}) = \sqrt{\frac{1}{3}[(a-d)^2 + (b-e)^2 + (c-f)^2]}. \quad (9)$$

Fuzzy MULTIMOORA Method

Fuzzy MULTIMOORA was introduced by Brauers et al. (2011). In this study, we employ the modified version as reported by Balezentiene et al. (2013). The fuzzy MULTIMOORA begins with fuzzy decision matrix $\tilde{X}$, where $\tilde{x}_{ij} = (x_{ij1}, x_{ij2}, x_{ij3})$ are aggregated responses of alternatives on objectives.

The Fuzzy Ratio System

The Ratio System defines normalization of the fuzzy numbers $\tilde{x}_{ij}$ resulting in matrix of dimensionless numbers. The normalization is performed by comparing appropriate values of fuzzy numbers:

$$\tilde{x}_{ij}^{*}=(x_{ij1}^{*},x_{ij2}^{*},x_{ij3}^{*})=\begin{cases}x_{ij1}^{*}=x_{ij1}\Big/\sqrt{\frac{1}{3}\sum_{i=1}^{m}\left[(x_{ij1})^{2}+(x_{ij2})^{2}+(x_{ij3})^{2}\right]}\\ x_{ij2}^{*}=x_{ij2}\Big/\sqrt{\frac{1}{3}\sum_{i=1}^{m}\left[(x_{ij1})^{2}+(x_{ij2})^{2}+(x_{ij3})^{2}\right]},\forall i,j.\\ x_{ij3}^{*}=x_{ij3}\Big/\sqrt{\frac{1}{3}\sum_{i=1}^{m}\left[(x_{ij1})^{2}+(x_{ij2})^{2}+(x_{ij3})^{2}\right]}\end{cases} \tag{10}$$

The normalization is followed by computation of the overall utility scores, $\tilde{y}_{i}^{*}$, for each i^{th} alternative. The normalized ratios are added or subtracted with respect to the type of criteria:

$$\tilde{y}_{i}^{*}=\sum_{j=1}^{g}\tilde{x}_{ij}^{*}\ominus\sum_{j=g+1}^{n}\tilde{x}_{ij}^{*}, \tag{11}$$

where $g=1,2,\ldots,n$ stands for number of criteria to be maximized. Then each ratio $\tilde{y}_{i}^{*}=(y_{i1}^{*},y_{i2}^{*},y_{i3}^{*})$ is defuzzified:

$$BNP_{i}=\frac{y_{i1}^{*}+y_{i2}^{*}+y_{i3}^{*}}{3} \tag{12}$$

BNP_{i} denotes the best non-fuzzy performance value of the i^{th} alternative. Consequently, the alternatives with higher BNP values are attributed with higher ranks.

The Fuzzy Reference Point

The fuzzy Reference Point approach is based on the fuzzy Ratio System. The Maximal Objective Reference Point (vector) $\tilde{r}$ is found according to ratios found in Eq. 10. The j^{th} coordinate of the reference point resembles the fuzzy maximum or minimum of the j^{th} criterion $\tilde{x}_{j}^{+}$, where

$$\begin{cases}\tilde{x}_{j}^{+}=\left(\max_{i}x_{ij1}^{*},\max_{i}x_{ij2}^{*},\max_{i}x_{ij3}^{*}\right),\ j\le g;\\ \tilde{x}_{j}^{+}=\left(\min_{i}x_{ij1}^{*},\min_{i}x_{ij2}^{*},\min_{i}x_{ij3}^{*}\right),\ j>g.\end{cases} \tag{13}$$

Then every element of normalized responses matrix is recalculated and final rank is given according to deviation from the reference point (Eq. 13) and the Min-Max Metric of Tchebycheff:

$$\min_i \left(\max_j d(\tilde{r}_j, \tilde{x}_{ij}^*) \right). \quad (14)$$

The Fuzzy Full Multiplicative Form

Overall utility of the i^{th} alternative can be expressed as a dimensionless number by employing Eq. 8:

$$\tilde{U}_i' = \tilde{A}_i \oslash \tilde{B}_i, \quad (15)$$

$\tilde{A}_i = (A_{i1}, A_{i2}, A_{i3}) = \prod_{j=1}^{g} \tilde{x}_{ij}$, $i = 1, 2, \cdots, m$ denotes the product of objectives of the i^{th} alternative to be maximized with $g = 1, \cdots, n$ being the number of criteria to be maximized.

$\tilde{B}_i = (B_{i1}, B_{i2}, B_{i3}) = \prod_{j=g+1}^{n} \tilde{x}_{ij}$ denotes the product of objectives of the i^{th} alternative to be minimized with $n - g$ as the number of criteria to be minimized. Since the overall utility $\tilde{U}_i'$ is a fuzzy number, one needs to defuzzify it to rank the alternatives (cf. Eq. 12). The higher the best non-fuzzy performance value (BNP), the higher will be the rank of a certain alternative.

Thus, the fuzzy MULTIMOORA summarizes fuzzy MOORA (i. e. fuzzy Ratio System and fuzzy Reference Point) and the fuzzy Full Multiplicative Form.

Employing this theory and as said before to each objective a minus value or a max value of 9% corresponding with the confidence level will be given. For instance, input of contractor a1 into criterion 6 being 12 is replaced in a fuzzy reasoning by 10.92, 12 and 13.08. A voter can give more importance to contractor a1 and to criterion 6 by preferring 13.8 above 12.

The three parts of Fuzzy MULTIMOORA presents the following results as given in table 9. The summary of the three parts is made by the Ordinal Dominance Theory as explained earlier.

Table 9. Ranking by Fuzzy MULTIMOORA after its three parts and with the application of Ordinal Dominance Theory (a)

	Fuzzy Ratio System	Fuzzy Reverence Point Method	Fuzzy Multiplicative Form	Fuzzy MULTIMOORA
a6	1	1	3	1
a1	2	3	2	2
a10	3	2	4	3
a4	4	5	1	4
a5	5	7	5	5
a3	6	8	6	6
a8	8	4	7	7
a14	7	11	8	8
a13	10	6	9	9
a9	9	13	10	10
a7	12	10	11	11
a11	13	9	12	12
a12	11	12	13	13
a2	14	14	14	14
a15	15	15	15	15

(a) Calculations available from the authors

Table 10 ranks the three possibilities for refining market analysis.

Table 10. Ranking Contractors after the three Possibilities (a)

MOORA with 18% spread		MULTIMOORA with 18% spread		Fuzzy MULTIMOORA no spread	
a6	1	a6	**1**	a6	**1**
a10	2	a4	**2**	a1	2
a1	3	a10	**3**	a10	3
a4	4	a1	**4**	a4	4
		a5	**5**	a5	5
		a3	**6**	a3	6
		a8	**7**	a8	7
		a14	**8**	a14	8
		a13	**9**	a13	9
		a9	**10**	a9	10

a7	**11**	a7	11
a11	**12**	a11	12
a12	**13**	a12	13
a2	**14**	a2	14
a15	**15**	a15	15

(a) Calculations available from the authors. To make it easier to understand MULTIMOORA or in particular to apply MULTIMOORA for marketing research the software is made in Excel style: first in numbers and then in control modus for formulas. For Excel applications, see: Herkenhoff and Fogli, 2013; Quirk, 2011.

Contractor a6 is preferred overall, which brings much certainty on this solution. Contrary to MULTIMOORA with 18% spread Contractor a1 is the second best as the method without spread shows its domination on the remaining other ones.

Nevertheless, one has to be aware about the real outcome. In the worst case, it could be that a client asks for a 9% additional effort from the side of the contractor. Can the winning contractor not anticipate this situation? Of course, he can, however with the danger that the winning contractor would become one of his colleagues.

On the other side, the contractor will be quasi certain that the client will buy his constructions, unless outside influences would interfere.

The theory is of general use each time a sample replaces total data mining around a certain phenomenon. Application on Gallup polls concerning public opinion, general elections in particular, form another example of information sampling.

5 Conclusion

The Belgian society called CIM is doing marketing research for all Belgian newspapers, magazines and cinema arriving at a spread of 24% as an average for all newspapers and even for some local newspapers at a spread of 30%, which is scientific nonsense but accepted by the publishers of advertisement. On the other side technical problems will ask for a much smaller standard deviation like for instance a standard error of 0.1% for the possibility that a dike is not strong enough for an eventual spring tide. Something in between the usual standard error for marketing research accepted is 5%.

Is it possible to avoid this Spread by Sampling? Here Multi-Objective Optimization Methods may be helpful with the additional question: which methods of MOO are useful in this case? It could not be methods based on the SAW principle as the choice of weights is another point of uncertainty. Neither

can be thought of methods comparing objectives or alternative solutions two by two with in this way being a victim of the Condorcet-Arrow Paradox. Rather have to be thought of methods based on dimensionless measurements like in the MOORA and MULTIMOORA Methods.

To the Ratio Method and the Reference Point Method of MOORA a third method is added in MULTIMOORA: The Full Multiplicative Form. The use of three different methods of MOO is more robust than using one or two.

Compared to crisp, fuzzy, interval-valued and intuitionistic fuzzy numbers, the neutrosophic set provides provide significantly greater flexibility, which can be conducive for solving decision-making problems associated with uncertainty, estimations and predictions.

Decision Making can be quantified by setting up a Decision Matrix with for instance Objectives or Criteria as columns and alternative solutions like Projects as rows. In this study Decision Making is quantified in its objectives, with the problem of normalization, due to the different units of the objectives and with the problem of importance. A MULTIMOORA method, chosen for its robustness instead of many other competing methods, will solve the problems of normalization and of importance, whereas Fuzzy MULTIMOORA will take care of the annoying spread in the samples.

Beside this method one has to be aware of the Universe around the sample, which is not directly quantitative. The Universe has not to be a disturbing factor.

It was Fuzzy MULTIMOORA which brought the solution to the Spread Problem by considering all the possible extreme positions delivered by the standard error. The outcome would have the form of an upside-down Gauss curve however not symmetrical but skewed and with the restriction that the solutions are not continuous but discrete.

Finally, a correction was made by the introduction of the Neutrosophic Extension of MULTIMOORA.

The example of disclosing the desiderata of potential buyers of property in Lithuania presents an illustration of the theory. However, the theory is of general use each time a sample replaces total mining of all data around a certain phenomenon like for Gallup polls concerning public opinion, general elections in particular.

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VIII

Multi-objective Geometric Programming Problem Based on Neutrosophic Geometric Programming Technique

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Abstract

The chapter aims to give computational algorithm to solve a multi-objective non-linear programming problem using Neutrosophic geometric programming technique. As the Neutrosophic optimization technique utilizes degrees of truth-membership, falsity-membership and indeterminacy-membership functions, we made a study of correspondence among those membership functions to see its impact on optimization. Also, we made a comparative study of optimal solution between intuitionistic fuzzy geometric programming and Neutrosophic geometric programming technique. The developed algorithm has been illustrated by a numerical example. Finally, an application of proposed Neutrosophic geometric programming technique on gravel box design problem is presented.

Keywords

Neutrosophic set, Single valued Neutrosophic set, Multi-objective non-linear programming, Neutrosophic geometric programming.

1 Introduction

The concept of fuzzy sets was introduced by Zadeh in 1965 [1]. Since the fuzzy sets and fuzzy logic have been applied in many real applications to handle uncertainty. The traditional fuzzy sets use one real value $\mu_A(x) \in [0, 1]$ to represents the truth membership function of a fuzzy set a defined on universe X. In some applications, we should consider not only the truth membership

supported by the evident but also the falsity membership against by the evident. That is beyond the scope of fuzzy sets and interval valued fuzzy sets. In 1986, Atanassov [3], [5] introduced the intuitionistic fuzzy sets which is a generalisation of fuzzy sets. The intuitionistic fuzzy sets consider both truth membership and falsity membership. Intuitionistic fuzzy sets can only handle incomplete information not the indeterminate information and inconsistent information. In IFS, sum of membership-degree and non-membership degree of a vague parameter is less than unity. Therefore, a certain amount of incomplete information or indeterminacy arises in an intuitionistic fuzzy set. It cannot handle all types of uncertainties successfully in different real physical problems. Hence further generalization of fuzzy set as well as intuitionistic fuzzy sets are required. In neutrosophic sets indeterminacy is quantified explicitly and truth membership, indeterminacy membership and falsity membership are independent. Neutrosophy was introduced by Florentin Smarandache in 1995 [4] which is actually generalization of different types of FS and IFS. The term "neutrosophy" means knowledge of neutral thought. This neutral concept makes the different between NS and other sets like FS, IFS. Modeling of most of real life problems involving optimization process turns out to be a multi-objective programming problem in a natural way. In this field, a paper named Multi-objective geometric programming problem with weighted-sum method by A.K. Ojha, A.K. Das has been published in the journal of computing 2010 [12]. In 1971 L.D. Paschal and A. Ben. Israel [16] developed a vector valued criteria in geometric programming. In 1978 a paper Fuzzy linear programming with several objective functions has been published by H.J Zimmermann [15]. In 1992 M.P. Bishal [13] and in 1990 R.k. Verma [14] has studied fuzzy programming technique to solve multi-objective geometric programming problems. In 2007 B. Jana and T.K. Roy [9] has studied multi-objective intuitionistic fuzzy linear programming problem and its application in Transportation model and in 2009 G.S. Mahapatra and T.K. Roy [10] developed multi-objective intuitionistic fuzzy mathematical programming problem and its application in Reliability optimization model. In this present study, a new approach of Neutrosophic Optimization (NO) is proposed. A multi-objective non-linear programming problem is solved by geometric programming technique.

2 Some Preliminaries

2.1 Definition -1 (Fuzzy set) [1]

Let X is a fixed set. A fuzzy set A of X is an object having the form $\tilde{A}$ = {(x,μ_A (x)), x ∈ X} where the function $\mu_A(x)$: X → [0, 1] defines the truth membership of the element x ∈ X to the set A.

2.2 Definition-2 (Intuitionistic fuzzy set) [3]

Let a set X be fixed. An intuitionistic fuzzy set or IFS $\tilde{A}^i$ in X is an object of the form $\tilde{A}^i = \{< X, \mu_A(x), \nu_A(x) > / x \in X\}$ where $\mu_A(x)$: X→ [0, 1] and $\nu_A(x)$: X→ [0, 1] define the Truth-membership and Falsity-membership respectively , for every element of x∈ X , $0 \leq \mu_A(x) + \nu_A(x) \leq 1$.

2.3 Definition-3 (Neutrosophic set) [4]

Let X be a space of points (objects) and $x \in X$. A neutrosophic set $\tilde{A}^n$ in X is defined by a Truth-membership function$\mu_A(x)$, an indeterminacy-membership function $\sigma_A(x)$ and a falsity-membership function $\nu_A(x)$ and having the form $\tilde{A}^n = \{< X, \mu_A(x), \sigma_A(x), \nu_A(x) > / x \in X\}$. $\mu_A(x)$, $\sigma_A(x)$ *and* $\nu_A(x)$ are real standard or non-standard subsets of

$]0^-, 1^+[$. that is

$$\mu_A(x) : X \rightarrow]0^-, 1^+[$$
$$\sigma_A(x) : X \rightarrow]0^-, 1^+[$$
$$\nu_A(x) : X \rightarrow]0^-, 1^+[$$

There is no restriction on the sum of $\mu_A(x)$, $\sigma_A(x)$ *and* $\nu_A(x)$, so

$0^- \leq \sup \mu_A(x) + \sup \sigma_A(x) + \sup \nu_A(x) \leq 3^+$

2.4 Definition-3 (Single valued Neutrosophic sets) [6]

Let X be a universe of discourse. A single valued neutrosophic set $\tilde{A}^n$ over X is an object having the form $\tilde{A}^n = \{< X, \mu_A(x), \sigma_A(x), \nu_A(x) > / x \in X\}$ where $\mu_A(x)$: X→ [0, 1], $\sigma_A(x)$: X→[0, 1] and $\nu_A(x)$: X→ [0, 1] with $0 \leq \mu_A(x) + \sigma_A(x) + \nu_A(x) \leq 3$ for all x ∈ X.

Example1: Assume that X = [x_1, x_2, x_3]. X_1 is capability, x_2 is trustworthiness and x_3 is price. The values of x_1, $x_2$and x_3 are in [0, 1]. They are obtained from the questionnaire of some domain experts, their option could be a degree of "good service", a degree of indeterminacy and a degree of "poor service". A is a single valued neutrosophic set of X defined by

$A = \langle 0.3,0.4,0.5\rangle/x_1 + \langle 0.5,0.2,0.3\rangle/x_2 + \langle 0.7,0.2,0.2\rangle/x_3$

2.5 Definition- 4(Complement): [6]

The complement of a single valued neutrosophic set A is denoted by c(A) and is defined by

$$\mu_{c(A)}(x) = \nu_A(x)$$
$$\sigma_{c(A)}(x) = 1 - \sigma_A(x)$$
$$\nu_{c(A)}(x) = \mu_A(x) \qquad \text{for all x in X.}$$

Example 2: let A be a single valued neutrosophic set defined in example 1. Then, c (A) = ⟨0.5,0.6,0.3⟩/x_1 + ⟨0.3,0.8,0.5⟩/x_2 +⟨0.2,0.8,0.7⟩/x_3.

2.6 Definition 5(Union):[6]

The union of two single valued neutrosophic sets A and B is a single valued neutrosophic set C, written as C = A ∪ B, whose truth-membership, indeterminacy-membership and falsity-membership functions are are given by

$\mu_c(x) = \max(\mu_A(x), \mu_B(x))$
$\sigma_c(x) = \max(\sigma_A(x), \sigma_B(x))$
$\nu_c(x) = \min(\nu_A(x), \nu_B(x))$ for all x in X

Example 3: Let A and B be two single valued neutrosophic sets defined in example -1. Then, A ∪ B = ⟨0.6,0.4,0.2⟩/x_1 + ⟨0.5,0.2,0.3⟩/x_2 + ⟨0.7,0.2,0.2⟩/x_3.

2.7 Definition 6(Intersection):[6]

The Intersection of two single valued neutrosophic sets A and B is a single valued neutrosophic set C, written as C = A ∩ B, whose truth-membership, indeterminacy-membership and falsity-membership functions are are given by

$\mu_c(x) = \min(\mu_A(x), \mu_B(x))$
$\sigma_c(x) = \min(\sigma_A(x), \sigma_B(x))$
$\nu_c(x) = \max(\nu_A(x), \nu_B(x))$ for all x in X

Example 4: Let A and B be two single valued neutrosophic sets defined in example -1. Then, A ∩ B = ⟨0.3,0.1,0.5⟩/x_1 + ⟨0.3,0.2,0.6⟩/x_2 + ⟨0.4,0.1,0.5⟩/x_3.

Here, we notice that by the definition of complement, union and intersection of single valued neutrosophic sets, single valued neutrosophic sets satisfy the most properties of classic set, fuzzy set and intuitionistic fuzzy set. Same as fuzzy set and intuitionistic fuzzy set, it does not satisfy the principle of middle exclude [17-21].

3 Multi-objective Geometric Programming Problem

A multi-objective geometric programming problem can be defined as

Find $x=(x_1,x_2,\ldots\ldots\ldots x_n)^T$,so as to (1)

Min $f_{k0}(x) = \sum_{t=1}^{T_{k0}} C_{k0t} \prod_{j=1}^{n} x_j^{a_{k0tj}}$ k=1,2,……,p

such that $f_i(x) = \sum_{t=1}^{T_i} c_{it} \prod_{j=1}^{n} x_j^{a_{itj}} \leq 1$ i=1 ,2, ,m

$x_j > 0$ j= 1,2, n

where $c_{k0t} > 0$ for all k and t . a_{itj} ,a_{k0tj} are all real ,for all i, k, t, j.

4 Computational Algorithm

Step 1: Solve the MONLP problem (1) as a single objective non-linear problem p times for each problem by taking one of the objectives at a time and ignoring the others. These solutions are known as ideal solutions. Let x^k be the respective optimal solution for the k[th] different objective and evaluate each objective value for all these k[th] optimal solution.

Step 2: From the result of step-1, determine the corresponding values for every objective for each derived solution. With the values of all objectives at each ideal solution, pay-off matrix can be formulated as follows.

$$\begin{bmatrix} f_1{}^*(x^1) & f_2(x^1) \dots \dots \dots \dots f_p(x^1) \\ f_1(x^2) & f_2{}^*(x^2) \dots \dots \dots \dots f_p(x^2) \\ \dots\dots\dots\

$$\sigma_k(f_k(x)) = \begin{cases} 1 & if\ f_k(x) \le {L_k}^{\sigma} \\ \frac{{U_k}^{\sigma} - f_k(x)}{{U_k}^{\sigma} - {L_k}^{\sigma}} & if\ {L_k}^{\sigma} \le f_k(x) \le {U_k}^{\sigma} \\ 0 & if\ f_k(x) \ge {U_k}^{\sigma} \end{cases},$$

$$\nu_k(f_k(x)) = \begin{cases} 0 & if\ f_k(x) \le {L_k}^{\nu} \\ \frac{f_k(x) - {L_k}^{\nu}}{{U_k}^{\nu} - {L_k}^{\nu}} & if\ {L_k}^{\nu} \le f_k(x) \le {U_k}^{\nu} \\ 1 & if\ f_k(x) \ge {U_k}^{\nu} \end{cases}$$

and $0 \le \mu_k(f_{k0}(x)) + \nu_k(f_{k0}(x)) + \sigma_k(f_k(x)) \le 3$
for k= 1, 2, ……,p.

Step 7. Now a Neutrosophic geometric programming technique for multi-objective non-linear programming problem with the linear Truth-membership, Falsity-membership and Indeterminacy functions can be written as

Maximize $(\mu_1(f_{10}(x)), \mu_2(f_{20}(x)),\ldots\ldots\ldots\ldots, \mu_p(f_{p0}(x)))$ (2)

Minimize $(\nu_1(f_{10}(x))\ \nu_2(f_{20}(x))\ \ldots\ldots\ldots\ldots, \nu_p(f_{p0}(x)))$

Maximize $(\sigma_1(f_{10}(x)), \sigma_2(f_{20}(x)),\ldots\ldots\ldots\ldots\ldots, \sigma_p(f_{p0}(x)))$

Subject to $f_i(x) = \sum_{t=1}^{T_i} c_{it} \prod_{j=1}^{n} x_j^{a_{itj}} \le 1$ for i=1, 2,……m

$x_j > 0$, j= 1,2,……n.

Using weighted sum method, the multi-objective non-linear programming problem (2) reduces to

Min $V_{MA}(x) = \sum_{k=1}^{p} w_k (\nu_k (f_{k0}(x)) - \mu_k(f_{k0}(x)) - \sigma_k(f_{k0}(x)))$ (3)

Min $V_{MA}(x) = \left(1 + \frac{1}{1-t} + \frac{1}{s}\right) \sum_{k=1}^{p} w_k \frac{\sum_{t=1}^{T_{k0}} C_{kot} \prod_{j=1}^{n} x_j^{a_{kotj}}}{{U_k}^{\mu} - {L_k}^{\mu}} - \left\{ \left((1 + \frac{1}{1-t} + \frac{1}{s}) \sum_{k=1}^{p} w_k \frac{{U_k}^{\mu}}{{U_k}^{\mu} - {L_k}^{\mu}} \right) - \frac{1}{s} \right\}$

Subject to $f_i(x) = \sum_{t=1}^{T_i} c_{it} \prod_{j=1}^{n} x_j^{a_{itj}} \le 1$ i=1, 2,……m

$x_j > 0$, j= 1,2,……n.

Excluding the constant term, the above (3) reduces to the following geometric programming problem

Min $V_{MA1}(x) = (1 + \frac{1}{1-t} + \frac{1}{s}) \sum_{k=1}^{p} w_k \frac{\sum_{t=1}^{T_{k0}} C_{kot} \prod_{j=1}^{n} x_j^{a_{kotj}}}{{U_k}^{\mu} - {L_k}^{\mu}}$ (4)

Such that $f_i(x) = \sum_{t=1}^{T_i} c_{it} \prod_{j=1}^{n} x_j^{a_{itj}} \le 1$ i=1, 2,……..m;

$x_j > 0$, j=1,2,……n.

Here t, s ϵ (0, 1) are pre-determined real numbers.

where $V_{MA}(f_{k0}(x))= V_{MA1}\ (f_{k0}(x))-\left\{ (1+\frac{1}{1-t}+\frac{1}{s})\ \left(\ \sum_{k=1}^{p} w_k\ \frac{{U_k}^{\mu}}{{U_k}^{\mu}-{L_k}^{\mu}}\ \right)-\frac{1}{s}\right\}$.

Here (4) is a posynomial geometric programming problem with

DD = $\sum_{k=1}^{p} T_{k0} + \sum_{i=1}^{m} T_i - n - 1$.

It can be solved by usual geometric programming technique.

Definition: Neutrosophic Pareto (or NS Pareto) optimal solution

A decision variable x^* Є X is said to be a NS Pareto optimal solution to the Neutrosophic GPP (2) if there does not exit another x Є X such that $\mu_k(f_{k0}(x)) \le \mu_k(f_{k0}(x^*))$, $\nu_k(f_{k0}(x)) \ge \nu_k(f_{k0}(x^*))$ and $\sigma_k(f_{k0}(x)) \le \sigma_k(f_{k0}(x^*))$ for all k=1,2,……..p. and $\mu_j(f_{j0}(x)) \ne \mu_j(f_{j0}(x^*))$, $\nu_j(f_{j0}(x)) \ne \nu_j(f_{j0}(x^*))$ and $\sigma_j(f_{j0}(x)) \ne \sigma_j(f_{j0}(x^*))$ for at least one j, j = 1,2,……..p.

Some basic theorems on M-N Pareto optimal solutions are introduced below.

Theorem 1 The solution of (2) based on weighted sum method Neutrosophic GP problem (3) is weakly NS Pareto optimal.

Proof. Let x^* ϵ X be a solution of the Neutrosophic GP problem. Let us suppose that it is not weakly M-N Pareto optimal. In this case there exit another x ϵ X such that $\mu_k(f_{k0}(x)) < \mu_k(f_{k0}(x^*))$, $\nu_k(f_{k0}(x)) > \nu_k(f_{k0}(x^*))$ and $\sigma_k(f_{k0}(x)) < \sigma_k(f_{k0}(x^*))$. for all k=1,2,……..p. Observing that $\mu_k(f_{k0}(x))$ is strictly monotone decreasing function with respect to $f_{k0}(x)$, this implies $\mu_k(f_{k0}(x)) > \mu_k(f_{k0}(x^*))$ and $\nu_k(f_{k0}(x))$ is strictly monotone increasing function with respect to $f_{k0}(x)$, this implies $\nu_k(f_{k0}(x)) < \nu_k(f_{k0}(x^*))$ and also $\sigma_k(f_{k0}(x)) > \sigma_k(f_{k0}(x^*))$. Thus we have $\sum_{k=1}^{p} w_k\ \mu_k(f_{k0}(x)) > \sum_{k=1}^{p} w_k\ \mu_k(f_{k0}(x^*))$, $\sum_{k=1}^{p} w_k\ \nu_k(f_{k0}(x)) < \sum_{k=1}^{p} w_k\ \nu_k(f_{k0}(x^*))$ and $\sum_{k=1}^{p} w_k\ \sigma_k(f_{k0}(x)) > \sum_{k=1}^{p} w_k\ \sigma_k(f_{k0}(x^*))$. This is a contradiction to the assumption that x^* is a solution of the Neutrosophic GP problem (2). Thus x^* is weakly NS Pareto optimal.

Theorem 2 The unique solution of Neutrosophic GP problem (3) based on weighted sum method is weakly NS Pareto optimal.

Proof. Let x^* Є X be a unique solution of the Neutrosophic GP problem. Let us suppose that it is not weakly NS Pareto optimal. In this case there exit another x Є X such that $\mu_k(f_{k0}(x)) \le \mu_k(f_{k0}(x^*))$, $\nu_k(f_{k0}(x)) \ge \nu_k(f_{k0}(x^*))$ for all k=1,2,……..p and $\mu_l(f_{l0}(x)) < \mu_l(f_{l0}(x^*))$, $\nu_l(f_{l0}(x)) > \nu_l(f_{l0}(x^*))$ for

at least one l. Observing that $\mu_k(f_{k0}(x))$ is strictly monotone decreasing function with respect to $f_{k0}(x)$, this implies $\mu_k(f_{k0}(x)) > \mu_k(f_{k0}(x^*))$ and $\nu_k(f_{k0}(x))$ is strictly monotone increasing function with respect to $f_{k0}(x)$, this implies $\nu_k(f_{k0}(x)) < \nu_k(f_{k0}(x^*))$ and also $\sigma_k(f_{k0}(x))$ is strictly monotone decreasing function with respect to $f_{k0}(x)$, this implies $\sigma_k(f_{k0}(x)) > \sigma_k(f_{k0}(x^*))$. Thus we have $\sum_{k=1}^{p} w_k\, \mu_k(f_{k0}(x)) \geq \sum_{k=1}^{p} w_k\, \mu_k(f_{k0}(x^*))$ and $\sum_{k=1}^{p} w_k\, \nu_k(f_{k0}(x)) \leq \sum_{k=1}^{p} w_k\, \nu_k(f_{k0}(x^*))$ and $\sum_{k=1}^{p} w_k\, \sigma_k(f_{k0}(x)) \geq \sum_{k=1}^{p} w_k\, \sigma_k(f_{k0}(x^*))$.

On the other hand, the uniqueness of x^* means that:

$\sum_{k=1}^{p} w_k\, \mu_k(f_{k0}(x^*)) < \sum_{k=1}^{p} w_k\, \mu_k(f_{k0}(x))$, $\sum_{k=1}^{p} w_k\, \nu_k(f_{k0}(x^*)) > \sum_{k=1}^{p} w_k\, \nu_k(f_{k0}(x))$ and $\sum_{k=1}^{p} w_k\, \sigma_k(f_{k0}(x^*)) < \sum_{k=1}^{p} w_k\, \sigma_k(f_{k0}(x))$.

The two sets inequalities above are contradictory and thus x^* is weakly NS Pareto optimal.

5 Illustrated Example

Min $f_1(x_1, x_2) = {x_1}^{-1}{x_2}^{-2}$

Min $f_2(x_1, x_2) = 2\,{x_1}^{-2}{x_2}^{-3}$

Such that $x_1 + x_2 \leq 1$

Here pay-off matrix is $\begin{bmatrix} 6.75 & 60.78 \\ 6.94 & 57.87 \end{bmatrix}$

Define truth-membership, falsity-membership and indeterminacy-membership functions are as follows:

$$\mu_1(f_1(x)) = \begin{cases} \frac{6.94 - {x_1}^{-1}{x_2}^{-2}}{0.19} & if\ 6.75 \leq {x_1}^{-1}{x_2}^{-2} \leq 6.94 \\ 1 & if\ {x_1}^{-1}{x_2}^{-2} \leq 6.75 \\ 0 & if\ {x_1}^{-1}{x_2}^{-2} \geq 6.94 \end{cases}$$

$$\mu_2(f_2(x)) = \begin{cases} \frac{60.78 - 2{x_1}^{-2}{x_2}^{-3}}{2.91} & if\ 57.87 \leq 2\,{x_1}^{-2}{x_2}^{-3} \leq 60.78 \\ 1 & if\ 2\,{x_1}^{-2}{x_2}^{-3} \leq 57.87 \\ 0 & if\ 2\,{x_1}^{-2}{x_2}^{-3} \geq 60.78 \end{cases}$$

$\nu_1(f_1(x)) = 1 - \frac{1}{1-t}\,\mu_1(f_1(x))$, and $\nu_2(f_2(x)) = 1 - \frac{1}{1-t}\,\mu_2(f_2(x))$

$\sigma_1(f_1(x)) = \frac{1}{s}\,\mu_1(f_1(x)) - \frac{1-s}{s}$, $\sigma_2(f_2(x)) = \frac{1}{s}\,\mu_2(f_2(x)) - \frac{1-s}{s}$

Table 1. Optimal values of primal, dual variables and objective functions from neutrosophic geometric programming problem for different weights.

Weights W_1, W_2	optimal dual variables $w_{01}^*, w_{02}^*, w_{11}^*, w_{12}^*$	optimal primal variables		optimal objectives		Sum of optimal objectives
		x_1^*	x_2^*	$f_1^*(x_1^*, x_2^*)$	$f_2^*(x_1^*, x_2^*)$	$f_1^*(x_1^*, x_2^*)$ $+f_2^*(x_1^*, x_2^*)$
0.5, 0.5	0.6491609, 0.3508391, 1.3508391, 2.3508391	0.3649261	0.6491609	6.794329	58.53371	65.32803
0.9, 0.1	0.9415706, 0.0584294, 1.0584294, 2.0584294	0.3395821	0.6604179	6.751768	60.21212	66.96388
0.1, 0.9	0.1745920, 0.8254080, 1.8254080, 2.8254080	0.3924920	0.6075080	6.903434	57.90451	64.80794

Table 2**.** Comparison of optimal solutions by IFGP and NSGP technique.

optimization techniques	optimal decision variables x_1^*, x_2^*	optimal objective functions $f_1^*(x_1^*, x_2^*)$, $f_2^*(x_1^*, x_2^*)$	sum of optimal objective functions $f_1^*(x_1^*, x_2^*)$ $+f_2^*(x_1^*, x_2^*)$
Intuitionistic Fuzzy Geometric Programming (IFGP)	0.36611, 0.63389	6.797678 58.58212	65.37980
proposed Neutrosophic Geometric Programming(NSGP)	0.3649261, 0.6491609	6.794329 58.53371	65.32803

In Table.2, it is seen that NSGP technique gives better optimal result than IFGP technique.

6 Application of Neutrosophic Optimization in Gravel box Design Problem

Gravel box problem: A total of 800 cubic-meters of gravel is to be ferried across a river on a barrage. A box (with an open top) is to be built for this purpose. After the entire gravel has been ferried, the box is to be discarded. The transport cost per round trip of barrage of box is Rs 1 and the cost of materials of the ends of the box are Rs20/m^2 and the cost of materials of other two sides and bottom are Rs 10/m^2 and Rs 80/m^2. Find the dimension of the box that is to be built for this purpose and the total optimal cost. Let length = x_1 m, width = x_2 m, height = x_3 m. The area of the end of the gravel box =x_2x_3 m^2. Area of the sides =x_1x_3 m^2. Area of the bottom =x_1x_2 m^2 .The volume of the gravel box=$x_1x_2x_3$ m^3. Transport cost: Rs $\frac{80}{x_1x_2x_3}$. Material cost: $40x_2x_3$. So, the multi-objective geometric programming problem is

$$\text{Min } g_{01} = \frac{80}{x_1x_2x_3} + 40x_2x_3$$

$$\text{Min } g_{02} = \frac{80}{x_1x_2x_3}$$

$$\text{Such that } x_1x_2 + 2x_1x_3 \leq 4.$$

Here pay-off matrix is $\begin{bmatrix} 95.24 & 63.78 \\ 120 & 40 \end{bmatrix}$

Table. 3: Comparison of optimal solutions by IFGP and NSGP technique.

Optimization techniques	Optimal Decision Variables x_1^*, x_2^*, x_3^*	Optimal Objective Functions g_{01}^*, g_{02}^*	Sum of optimal objective values
Intuitionistic fuzzy geometric programming (IFGP)	1.2513842, 1.5982302, 0.7991151	101.1421624 50.0553670	151.1975294
Proposed neutrosophic geometric programming(NSGP)	1.2513843, 1.5982300, 0.7991150	101.1421582 50.0553655	151.1975237

7 Conclusion

In view of comparing the Neutrosophic geometric programming technique with Intuitionistic fuzzy geometric programming technique, we also obtained the solution of the undertaken numerical problem by Intuitionistic fuzzy optimization method and took the best result obtained for comparison with present study.

The objectives of the present study are to give the effective algorithm for Neutrosophic geometric programming method for getting optimal solutions to a multi-objective non-linear programming problem. Further the comparisons of results obtained for the undertaken problem clearly show the superiority of Neutrosophic geometric programming technique over Intuitionistic fuzzy geometric programming technique.

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IX

Using Neutrosophic Sets to Obtain PERT Three-Times Estimates in Project Management

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Abstract

Neutrosophic sets have been introduced as a generalization of crisp sets, fuzzy sets, and intuitionistic fuzzy sets to represent uncertain, inconsistent and incomplete information about real world problems. Elements of neutrosophic set are characterized by a truth-membership, falsity-membership and indeterminacy membership functions. For the first time, this chapter attempts to introduce the mathematical representation of Program Evaluation and Review Technique (PERT) in neutrosophic environment. Here the elements of three-times estimates of PERT are considered as neutrosophic elements. Score and accuracy functions are used to obtain crisp model of problem. The proposed method has been demonstrated by a suitable numerical example.

Keywords

Neutrosophic Sets, Project, Project Management, Gantt chart, CPM, PERT, Three-Time Estimate.

1 Introduction

A project is a one-time job that has a definite starting and ending dates, a clearly specified objective, a scope of work to be performed and a predefined budget. Each part of the project has an effect on overall project execution time,

so project completion on time depends on rightly scheduled plan. The main problem here is wrongly calculated activity durations due to lack of knowledge and experience. Lewis [1] defines project management as "the planning, scheduling and controlling of project activities to achieve project objectives-performance, cost and time for a given scope of work". The most popularly used techniques for project management are Gantt chart, Program Evaluation and Review Technique (PERT) and Critical Path Method (CPM). Gantt chart is an early technique of planning and controlling projects. Gantt charts are simple to construct, easy to understand and change. They can show plan and actual progress. However, it does not show interrelationships of activities. To overcome the limitation of Gantt chart, two project planning techniques-PERT and CPM were developed in 1950s. Both use a network and graphical model of a project, showing the activities, their interrelationships and starting and ending dates. In case of CPM, activity time can be estimated accurately and it does not vary much. In recent years, by depending on the fuzzy set theory for managing projects there were different PERT methods. However, the existing methods of fuzzy PERT have some drawbacks [2]:

– Cannot find a critical path in a fuzzy project network.

– The increasing of the possible critical paths, which is the higher risk path.

– Can't determine indeterminacy, which exist in real life situations.

In case of PERT, time estimates vary significantly [3][4]. Here three-time estimates which are optimistic(a), pessimistic(b) and most likely(m) are used. In practice, a question often arises as to how obtain good estimates of a, b, and m. The person who responsible for determining values of a, b, and m often face real problem due to uncertain, inconsistent, and incomplete information about real world. It is obvious that neutrosophic set theory is more appropriate than fuzzy set in modeling uncertainty that is associated with parameters such as activity duration time and resource availability in PERT. By using neutrosophic set theory in PERT technique, we can also overcome the drawbacks of fuzzy PERT methods. This chapter is organized as follows: In section 2, the basic concepts neutrosophic sets are briefly reviewed. In section 3, the mathematical model of neutrosophic PERT and the proposed algorithm is presented. In section 4, a suitable numerical example is illustrated. Finally, section 5 concludes the chapter with future work.

2 Preliminaries

In this section, the basic definitions involving neutrosophic set, single valued neutrosophic sets, trapezoidal neutrosophic numbers and operations on trapezoidal neutrosophic numbers are outlined.

Definition 1. [5] Let X be a space of points (objects) and $x \in X$. A neutrosophic set A in X is defined by a truth-membership function $T_A(x)$, an indeterminacy-membership function $I_A(x)$ and a falsity-membership function $F_A(x)$. $T_A(x)$, $I_A(x)$ and $F_A(x)$ are real standard or real nonstandard subsets of $]^-0, 1^+[$. That is $T_A(x): X \to]^-0, 1^+[$, $I_A(x): X \to]^-0, 1^+[$ and $F_A(x): X \to]^-0, 1^+[$. There is no restriction on the sum of $T_A(x)$, $I_A(x)$ and $F_A(x)$, so $0^- \leq \sup T_A(x) + \sup I_A(x) + \sup F_A(x) \leq 3^+$.

Definition 2. [5] Let X be a universe of discourse. A single valued neutrosophic set A over X is an object having the form $A = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle : x \in X\}$, where $T_A(x): X \to [0,1]$, $I_A(x): X \to [0,1]$ and $F_A(x): X \to [0,1]$ with $0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$ for all $x \in X$. The intervals $T_A(x)$, $I_A(x)$ and $F_A(x)$ denote the truth-membership degree, the indeterminacy-membership degree and the falsity membership degree of x to A, respectively. For convenience, a SVN number is denoted by $A = (a,b,c)$, where $a,b,c \in [0,1]$ and $a+b+c \leq 3$.

Definition 3. [6] Let $\alpha_{\tilde{a}}, \theta_{\tilde{a}}, \beta_{\tilde{a}} \in [0,1]$ and $a_1, a_2, a_3, a_4 \in \boldsymbol{R}$ such that $a_1 \leq a_2 \leq a_3 \leq a_4$. Then a single valued trapezoidal neutrosophic number, $\tilde{a} = \langle (a_1, a_2, a_3, a_4); \alpha_{\tilde{a}}, \theta_{\tilde{a}}, \beta_{\tilde{a}} \rangle$ is a special neutrosophic set on the real line set $\boldsymbol{R}$, whose truth-membership, indeterminacy-membership, and falsity-membership functions are given as follows [8]:

$$T_{\tilde{a}}(x) = \begin{cases} \alpha_{\tilde{a}} \left(\frac{x - a_1}{a_2 - a_1} \right) & if\ a_1 \leq x \leq a_2 \\ \alpha_{\tilde{a}} & if\ a_2 \leq x \leq a_3 \\ \alpha_{\tilde{a}} \left(\frac{a_4 - x}{a_4 - a_3} \right) & if\ a_3 < x \leq a_4 \\ 0 & otherwise \end{cases} \quad (1)$$

$$I_{\tilde{a}}(x) = \begin{cases} \frac{(a_2 - x + \theta_{\tilde{a}}(x - a_1))}{(a_2 - a_1)} & if\ a_1 \leq x \leq a_2 \\ \theta_{\tilde{a}} & if\ a_2 \leq x \leq a_3 \\ \frac{(x - a_3 + \theta_{\tilde{a}}(a_4 - x))}{(a_4 - a_3)} & if\ a_3 < x \leq a_4 \\ 1 & otherwise \end{cases} \quad (2)$$

$$F_{\tilde{a}}(x)=\begin{cases}\dfrac{\left(a_2-x+\beta_{\tilde{a}}(x-a_1)\right)}{(a_2-a_1)} & if\ a_1\le x\le a_2\\ \beta_{\tilde{a}} & if\ \ a_2\le x\le a_3\\ \dfrac{(x-a_3+\beta_{\tilde{a}}\,(a_4-x))}{(a_4-a_3)} & if\ a_3<x\le a_4\\ 1 & otherwise\end{cases} \tag{3}$$

where $\alpha_{\tilde{a}}$,$\theta_{\tilde{a}}$, $\beta_{\tilde{a}}$ denote the maximum truth-membership degree, minimum indeterminacy-membership degree and minimum falsity-membership degree respectively. A single valued trapezoidal neutrosophic number $\tilde{a}$=$\langle(a_1,a_2,a_3,a_4);\alpha_{\tilde{a}}\,,\theta_{\tilde{a}},\beta_{\tilde{a}}\rangle$ may express an ill-defined quantity about a,which is approximately equal to $[a_2,a_3]$.

Definition 4. [7] Let $\tilde{a}$=$\langle(a_1,a_2,a_3,a_4);\alpha_{\tilde{a}}\,,\theta_{\tilde{a}},\beta_{\tilde{a}}\rangle$ and $\tilde{b}$ = $\langle(b_1,b_2,b_3,b_4\,);\ \alpha_{\tilde{b}}\,,\theta_{\tilde{b}},\beta_{\tilde{b}}\rangle$ be two single valued trapezoidal neutrosophic numbers and $\gamma\neq o$ be any real number [9]. Then,

1. $\tilde{a}$ + $\tilde{b}$ =$\langle(a_1+b_1\,,a_2+b_2,a_3+b_3\,,a_4+b_4);\ \alpha_{\tilde{a}}\wedge\alpha_{\tilde{b}},\ \theta_{\tilde{a}}\vee\theta_{\tilde{b}},\ \beta_{\tilde{a}}\vee\beta_{\tilde{b}}\rangle$
2. $\tilde{a}$ - $\tilde{b}$ =$\langle(a_1-b_4\,,a_2-b_3,a_3-b_2\,,a_4-b_1);\ \alpha_{\tilde{a}}\wedge\alpha_{\tilde{b}},\ \theta_{\tilde{a}}\vee\theta_{\tilde{b}},\ \beta_{\tilde{a}}\vee\beta_{\tilde{b}}\rangle$
3. $\tilde{a}\tilde{b}=$
$$\begin{cases}\langle(a_1b_1\,,a_2b_2,a_3b_3,a_4b_4);\ \alpha_{\tilde{a}}\wedge\alpha_{\tilde{b}},\ \theta_{\tilde{a}}\vee\theta_{\tilde{b}},\ \beta_{\tilde{a}}\vee\beta_{\tilde{b}}\rangle & if\ (a_4>0,b_4>0)\\ \langle(a_1b_4\,,a_2b_3,a_3b_2,a_4b_1);\ \alpha_{\tilde{a}}\wedge\alpha_{\tilde{b}},\ \theta_{\tilde{a}}\vee\theta_{\tilde{b}},\ \beta_{\tilde{a}}\vee\beta_{\tilde{b}}\rangle & if\ (a_4<0,b_4>0\,)\\ \langle(a_4b_4\,,a_3b_3,a_2b_2,a_1b_1);\ \alpha_{\tilde{a}}\wedge\alpha_{\tilde{b}},\ \theta_{\tilde{a}}\vee\theta_{\tilde{b}},\ \beta_{\tilde{a}}\vee\beta_{\tilde{b}}\rangle & if\ (a_4<0,b_4<0\,)\end{cases}$$
4. $\frac{\tilde{a}}{\tilde{b}}=$
$$\begin{cases}\langle\left(\frac{a_1}{b_4},\frac{a_2}{b_3},\frac{a_3}{b_2},\frac{a_4}{b_1}\right);\ \alpha_{\tilde{a}}\wedge\alpha_{\tilde{b}},\ \theta_{\tilde{a}}\vee\theta_{\tilde{b}},\ \beta_{\tilde{a}}\vee\beta_{\tilde{b}}\rangle & if(a_4>0,b_4>0)\\ \langle\left(\frac{a_4}{b_4},\frac{a_3}{b_3},\frac{a_2}{b_2},\frac{a_1}{b_1}\right);\ \alpha_{\tilde{a}}\wedge\alpha_{\tilde{b}},\ \theta_{\tilde{a}}\vee\theta_{\tilde{b}},\ \beta_{\tilde{a}}\vee\beta_{\tilde{b}}\rangle & if(a_4<0,b_4>0\,)\\ \langle\left(\frac{a_4}{b_1},\frac{a_3}{b_2},\frac{a_2}{b_3},\frac{a_1}{b_4}\right);\ \alpha_{\tilde{a}}\wedge\alpha_{\tilde{b}},\ \theta_{\tilde{a}}\vee\theta_{\tilde{b}},\ \beta_{\tilde{a}}\vee\beta_{\tilde{b}}\rangle & if(a_4<0,b_4<0)\end{cases}$$
5. $\gamma\tilde{a}=\begin{cases}\langle(\gamma a_1,\gamma a_2,\gamma a_3,\gamma a_4);\alpha_{\tilde{a}}\,,\theta_{\tilde{a}}\,,\beta_{\tilde{a}}\rangle & if(\gamma>0)\\ \langle(\gamma a_4,\gamma a_3,\gamma a_2,\gamma a_1);\alpha_{\tilde{a}}\,,\theta_{\tilde{a}}\,,\beta_{\tilde{a}}\rangle & if\ (\gamma<0)\end{cases}$
6. $\tilde{a}^{-1}=\langle\left(\frac{1}{a_4},\frac{1}{a_3},\frac{1}{a_2},\frac{1}{a_1}\right);\alpha_{\tilde{a}}\,,\theta_{\tilde{a}}\,,\beta_{\tilde{a}}\rangle$, where $(\tilde{a}\neq 0)$.

3 PERT in Neutrosophic Environment and the Proposed Model

Like CPM, PERT uses network model. However, PERT has been traditionally used in new projects which have large uncertainty in respect of design, technology and construction. To take care of associated uncertainties, we

adopt neutrosophic environment for PERT activity duration.

The three-time estimates for activity duration are:

1. Optimistic time ($\tilde{a}$): it is the minimum time needed to complete the activity if everything goes well.
2. Pessimistic time($\tilde{b}$) : it is the maximum time needed to complete the activity if one encounters problems at every turn.
3. Most likely time, i.e. Mode ($\tilde{m}$): it is the time required to complete the activity in normal circumstances.

Where $\tilde{a},\tilde{b},\tilde{m}$ are trapezoidal neutrosophic numbers.

Based on three time estimates ($\tilde{a}$, $\tilde{b},\tilde{m}$), expected time and standard deviation of each activity should be calculated , and to do this we should first obtain crisp values of three time estimates.

To obtain crisp values of three-time estimates, we should use score functions and accuracy functions as follows:

Let $\tilde{a}$=< (a_1, a_2, a_3, a_4); $\alpha_{\tilde{a}}$,$\theta_{\tilde{a}}$, $\beta_{\tilde{a}}$ > be a single valued trapezoidal neutrosophic number; then

1. score function $S(\tilde{a}) = \left(\frac{1}{16}\right)[a_1 + a_2 + a_3 + a_4]\times\left[\alpha_{\tilde{a}} + (1 - \theta_{\tilde{a}}) + \left(1 - \beta_{\tilde{a}}\right)\right]$; (4)
2. accuracy function $A(\tilde{a}) = \left(\frac{1}{16}\right)[a_1 + a_2 + a_3 + a_4]\times\left[\alpha_{\tilde{a}} + (1 - \theta_{\tilde{a}}) + \left(1 + \beta_{\tilde{a}}\right)\right]$. (5)

After obtaining crisp values of each time estimate by using score function, the expected time and standard deviation of each activity calculated as follows;

$$T_{ij} = \frac{a+4m+b}{6} \quad (6)$$

and

$$\sigma_{ij} = \frac{b-a}{6} \quad (7)$$

where a, m, b are crisp values of optimistic, most likely and pessimistic time respectively,

T_{ij}= Expected time of ij activity and

σ_{ij} = Standard deviation of ij activity.

Once the expected time and standard deviation of each activity are calculated, PERT network is treated like CPM network for the purpose of calculation of network parameters like earliest/latest occurrence time of activity, critical path and floats.

Let a network N= $\langle E, A, T\rangle$, being a project model, is given. E is asset of events (nodes) and A$\subset E\times E$ is a set of activities. The set E=$\{1,2,\dots,n\}$ is

labeled in such a way that the following condition holds: $(i, j) \in A$ and $i < j$. The activity times in the network are determined by $T_{ij.}$

Notations of network solution and its calculations as follows:

T_i^e=Earliest occurrence time of predecessor event *i*,

T_i^l= Latest occurrence time of predecessor event *i*,

T_j^e=Earliest occurrence time of successor event *j*,

T_j^l= Latest occurrence time of successor event *j*,

T_{ij}^e/Start= Earliest start time of an activity ij,

T_{ij}^e/Finisht=Earliest finish time of an activityij,

T_{ij}^l/Start=Latest start time of an T_i^l activity ij,

T_{ij}^l/Finisht= Latest finish time of an activity ij,

$T_{ij} =$ Duration time of activity ij,

Earliest and Latest occurrence time of an event:

T_j^e=maximum $(T_j^e + T_{ij})$, calculate all T_j^e for jth event, select maximum value.

T_i^l=minimum $(T_j^l - T_{ij})$, calculate all T_i^l for ith event, select minimum value.

T_{ij}^e/Start=T_i^e,

T_{ij}^e/Finisht=$T_i^e + T_{ij}$,

T_{ij}^l/Finisht=T_j^l,

T_{ij}^l/Start=$T_j^l - T_{ij}$,

Critical path is the longest path in the network. At critical path, T_i^e=T_i^l, for all *i*.

Slack or Float is cushion available on event/ activity by which it can be delayed without affecting the project completion time.

Slack for ith event = $T_i^l - T_i^e$, for events on critical path, slack is zero.

The expected time of critical path (μ) and its variance (σ^2)calculated as follows;

$\mu = \sum T_{ij,}$ for all ij on critical path.

$\sigma^2 = \sum \sigma_{ij}^2$, for all ij on critical path.

From the previous steps we can conclude the proposed algorithm as follows:

1. To deal with uncertain, inconsistent and incomplete information about activity time, we considered three-time estimates of PERT technique as a single valued trapezoidal neutrosophic numbers.
2. Calculate membership functions of each single valued trapezoidal neutrosophic number, using equation 1, 2 and 3.
3. Obtain crisp model of PERT three-time estimates using score function equation as we illustrated previously.
4. Use crisp values of three time estimates to calculate expected time and standard deviation of each activity.
5. Draw PERT network diagram.
6. Determine floats and critical path, which is the longest path in network as we illustrated previously with details.
7. Calculate expected time and variance of critical path.
8. Determine expected project completion time.

4 Illustrative Example

Let us consider neutrosophic PERT and try to obtain crisp model from it. Since you are given the following data for a project:

Table 1. Input data for neutrosophic PERT.

Activity	Immediate Predecessors	Time (days)		
		$\tilde{a}$	$\tilde{m}$	$\tilde{b}$
A	------	$\tilde{1}$	$\tilde{3}$	$\tilde{5}$
B	------	$\tilde{2}$	$\tilde{4}$	$\tilde{6}$
C	A	$\tilde{7}$	$\tilde{9}$	$\widetilde{15}$
D	A	$\tilde{8}$	$\widetilde{10}$	$\widetilde{16}$
E	B	$\tilde{6}$	$\tilde{9}$	$\widetilde{14}$
F	C,D	$\widetilde{11}$	$\widetilde{15}$	$\widetilde{17}$
G	D,E	$\tilde{6}$	$\widetilde{11}$	$\widetilde{19}$
H	F,G	$\tilde{8}$	$\widetilde{12}$	$\widetilde{20}$

In the previous table $\tilde{a}$, $\tilde{m}$ and $\tilde{b}$ are optimistic, most likely and pessimistic time in neutrosophic environment, and considered as a single valued trapezoidal neutrosophic numbers.

Let

$\tilde{1} = \langle(0,2,4,5); 0.8,0.6,0.4\rangle$, $\tilde{2} = \langle(1,3,5,6);\ \ 0.2,0.3,0.5\rangle$,

$\tilde{3} = \langle(1,2,5,6); 0.2,0.5,0.6\rangle$,$\tilde{4} = \langle(1,2,5,7); 0.5,0.4,0.9\rangle$,

$\tilde{5} = \langle(2,4,7,10); 0.8,0.2,0.4\rangle$, $\tilde{6} = \langle(3,7,9,12); 0.7,0.2,0.5\rangle$, $\tilde{7} = \langle(5,8,9,13); 0.4,0.6,0.8\rangle$, $\tilde{8} = \langle(1,6,10,13); 0.9,0.1,0.3\rangle$,

$\tilde{9} = \langle(6,8,10,15); 0.6,0.4,0.7\rangle$, $\widetilde{10} = \langle(1,6,11,15); 0.7,0.6,0.3\rangle$, $\widetilde{11} = \langle(5,8,15,20); 0.8,0.2,0.5\rangle$, $\widetilde{12} = \langle(4,8,17,25); 0.3,0.6,0.4\rangle$,

$\widetilde{14} = \langle(7, 10,19,30); 0.8,0.4,0.7\rangle, \widetilde{15} = \langle(8, 10,20,35); 0.5,0.2,0.4\rangle,$
$\widetilde{16} = \langle(5, 15,25,30); 0.7,0.5,0.6\rangle, \widetilde{17} = \langle(10, 15,20,25); 0.2,0.4,0.6\rangle,$
$\widetilde{19} = \langle(15, 17,23,25); 0.9,0.7,0.8\rangle, \widetilde{20} = \langle(10, 12,27,30); 0.2,0.3,0.5\rangle.$

Step 1: To obtain crisp values of each single valued trapezoidal neutrosophic number, we should calculate score function as follows:

Score function $S(\tilde{1}) = \left(\frac{1}{16}\right)[0 + 2 + 4 + 5]\times[0.8 + (1 - 0.6) + (1 - 0.4)] = 1.24$

Score function $S(\tilde{2}) = \left(\frac{1}{16}\right)[1 + 3 + 5 + 6]\times[0.2 + (1 - 0.3) + (1 - 0.5)] = 1.31$

Score function $S(\tilde{3}) = \left(\frac{1}{16}\right)[1 + 2 + 5 + 6]\times[0.2 + (1 - 0.5) + (1 - 0.6)] = 0.96$

Score function $S(\tilde{4}) = \left(\frac{1}{16}\right)[1 + 2 + 5 + 7]\times[0.5 + (1 - 0.4) + (1 - 0.9)] = 1.12$

Score function $S(\tilde{5}) = \left(\frac{1}{16}\right)[2 + 4 + 7 + 10]\times[0.8 + (1 - 0.2) + (1 - 0.4)] = 3.16$

Score function $S(\tilde{6}) = \left(\frac{1}{16}\right)[3 + 7 + 9 + 12]\times[0.7 + (1 - 0.2) + (1 - 0.5)] = 3.87$

Score function $S(\tilde{7}) = \left(\frac{1}{16}\right)[5 + 8 + 9 + 13]\times[0.4 + (1 - 0.6) + (1 - 0.8)] = 2.19$

Score function $S(\tilde{8}) = \left(\frac{1}{16}\right)[1 + 6 + 10 + 13]\times[0.9 + (1 - 0.1) + (1 - 0.3)] = 4.68$

Score function $S(\tilde{9}) = \left(\frac{1}{16}\right)[6 + 8 + 10 + 15]\times[0.6 + (1 - 0.4) + (1 - 0.7)] = 3.66$

Score function $S(\widetilde{10}) = \left(\frac{1}{16}\right)[1 + 6 + 11 + 15]\times[0.7 + (1 - 0.6) + (1 - 0.3)] = 3.71$

Score function $S(\widetilde{11}) = \left(\frac{1}{16}\right)[5 + 8 + 15 + 20]\times[0.8 + (1 - 0.2) + (1 - 0.5)] = 6.3$

Score function $S(\widetilde{12}) = \left(\frac{1}{16}\right)[4 + 8 + 17 + 25]\times[0.3 + (1 - 0.6) + (1 - 0.4)] = 4.39$

Score function $S(\widetilde{14}) = \left(\frac{1}{16}\right)[7 + 10 + 19 + 30]\times[0.8 + (1 - 0.4) + (1 - 0.7)] = 7.01$

Score function $S(\widetilde{15}) = \left(\frac{1}{16}\right)[8 + 10 + 20 + 35]\times[0.5 + (1 - 0.2) + (1 - 0.4)] = 8.67$

Score function $S(\widetilde{16}) = \left(\frac{1}{16}\right)[5 + 15 + 25 + 30]\times[0.7 + (1 - 0.5) + (1 - 0.6)] = 7.5$

Score function $S(\widetilde{17}) = \left(\frac{1}{16}\right)[10 + 15 + 20 + 25]\times[0.2 + (1 - 0.4) + (1 - 0.6)] = 5.25$

Score function $S(\widetilde{19}) = \left(\frac{1}{16}\right)[15 + 17 + 23 + 25]\times[0.9 + (1 - 0.7) + (1 - 0.8)] = 7$

Score function $S(\widetilde{20}) = \left(\frac{1}{16}\right)[10 + 12 + 27 + 30]\times[0.2 + (1 - 0.3) + (1 - 0.5)] = 6.91$

Step 2: By putting score functions values as crisp values of each time estimate, we can calculate the expected time and variance of each activity as we illustrated with equations in the previous section. The expected time of each activity has been calculated and presented in table 2.

Table2**.** The expected time of each activity in the project.

Activity	Immediate Predecessors	Expected Time(days)
A	------	1
B	------	2
C	A	3
D	A	4
E	B	4
F	C,D	8
G	D,E	6
H	F,G	5

Step 3: Draw the network diagram by using Microsoft Project 2010.

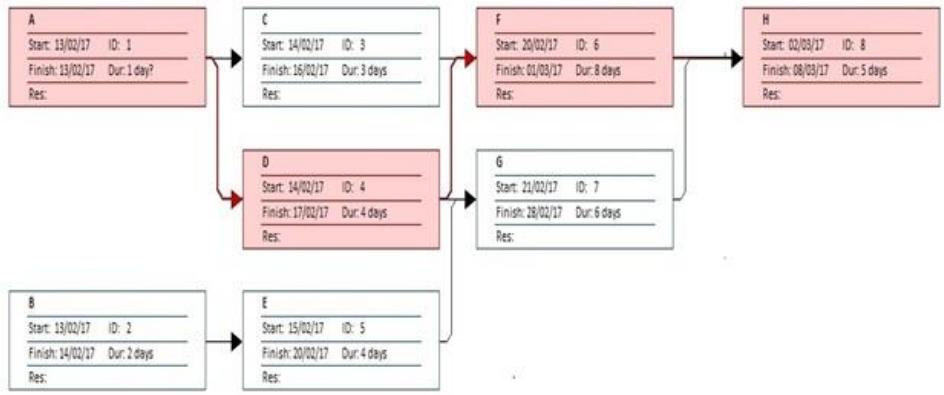

Fig. 1. Network of activities with critical path

From figure 1, we find that the critical path is A-D-F-H and is denoted by red line. The expected project completion time = $t_A + t_D + t_F + t_H$ = 18 days.

5 Conclusion

Neutrosophic set is a generalization of classical set, fuzzy set and intuitionistic fuzzy set because it not only considers the truth-membership and falsity- membership, but also an indeterminacy membership which is very obvious in real life situations. In this chapter, we have considered the three-time estimates of PERT as a single valued trapezoidal neutrosophic numbers and we used score function to obtain crisp values of three-time estimates. In future, the research will be extended to deal with different project management techniques.

Acknowledgements

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X

A Critical Path Problem Using Triangular Neutrosophic Number

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Abstract

The Critical Path Method (CPM) is one of several related techniques for planning and managing of complicated projects in real world applications. In many situations, the data obtained for decision makers are only approximate, which gives rise of neutrosophic critical path problem. In this chapter, the proposed method has been made to find the critical path in network diagram, whose activity time uncertain. The vague parameters in the network are represented by triangular neutrosophic numbers, instead of crisp numbers. At the end of the chapter, two illustrative examples are provided to validate the proposed approach**.**

Keywords

Neutrosophic Sets; Project Management; CPM; Score and Accuracy Functions.

1 Introduction

Project management is concerned with selecting, planning, execution and control of projects in order to meet or exceed stakeholders' need or expectation from project. Two techniques of project management, namely Critical Path Method (CPM) and Program Evaluation and Review Technique (PERT) where

developed in 1950s. [1] The successful implementation of CPM requires clear determined time duration for each activity.

Steps involved in CPM include [2]:

- Develop Work Breakdown Structure of a project, estimate the resources needed and establish precedence relationship among activities.
- Translate the activities into network.
- Carry out network computation and prepare schedule of the activities.

In CPM, the main problem is wrongly calculated activity durations, of large projects that have many activities. The planned value of activity duration time may change under certain circumstances and may not be presented in a precise manner due to the error of the measuring technique or instruments etc. It has been obvious that neutrosophic set theory is more appropriate to model uncertainty that is associated with parameters such as activity duration time and resource availability in CPM.

This chapter is organized as follows: In section 2, the basic concepts neutrosophic sets are briefly reviewed. In section 3, the mathematical model of neutrosophic CPM and the proposed algorithm is presented. In section 4, two numerical examples are illustrated. Finally, section 5 concludes the chapter with future work.

2 Preliminaries

In this section, the basic definitions involving neutrosophic set, single valued neutrosophic sets, triangular neutrosophic numbers and operations on triangular neutrosophic numbers are outlined.

Definition 1. [3, 5-7] Let X be a space of points (objects) and $x \in X$. A neutrosophic set A in X is defined by a truth-membership function $T_A(x)$, an indeterminacy-membership function $I_A(x)$ and a falsity-membership function $F_A(x)$. $T_A(x)$, $I_A(x)$ and $F_A(x)$ are real standard or real nonstandard subsets of $]^-0, 1^+[$. That is $T_A(x): X \to]^-0, 1^+[$, $I_A(x): X \to]^-0, 1^+[$ and $F_A(x): X \to]^-0, 1^+[$. There is no restriction on the sum of $T_A(x)$, $I_A(x)$ and $F_A(x)$, so $0^- \leq \sup T_A(x) + \sup I_A(x) + \sup F_A(x) \leq 3^+$.

Definition 2. [3, 8] Let X be a universe of discourse. A single valued neutrosophic set A over X is an object having the form $A=\{\langle x, T_A(x), I_A(x), F_A(x)\rangle : x \in X\}$, where $T_A(x): X \to [0,1]$, $I_A(x): X \to [0,1]$ and $F_A(x): X \to [0,1]$ with $0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$ for all $x \in X$. The intervals $T_A(x)$, $I_A(x)$ and $F_A(x)$ denote the truth-membership degree, the indeterminacy-membership degree and the

falsity membership degree of x to A, respectively. For convenience, a SVN number is denoted by A= (a, b, c), where a, b, $c\in$ [0, 1] and a+b+c≤3.

Definition 3. [4, 5] Let $\alpha_{\tilde{a}}, \theta_{\tilde{a}}, \beta_{\tilde{a}} \boldsymbol{\epsilon}[\mathbf{0}, \mathbf{1}]$ and $\boldsymbol{a_1}, \boldsymbol{a_2}, \boldsymbol{a_3} \boldsymbol{\epsilon}\, \boldsymbol{R}$ such that $\boldsymbol{a_1} \leq \boldsymbol{a_2} \leq \boldsymbol{a_3}$. Then a single valued triangular neutrosophic number, $\tilde{\boldsymbol{a}}$=$\langle(\boldsymbol{a_1}, \boldsymbol{a_2}, \boldsymbol{a_3}); \alpha_{\tilde{a}}, \theta_{\tilde{a}}, \beta_{\tilde{a}}\rangle$ is a special neutrosophic set on the real line set $\boldsymbol{R}$, whose truth-membership, indeterminacy-membership, and falsity-membership functions are given as follows:

$$T_{\tilde{a}}(x) = \begin{cases} \alpha_{\tilde{a}}\left(\dfrac{x-a_1}{a_2-a_1}\right) & if\ a_1 \leq x \leq a_2 \\ \alpha_{\tilde{a}} & if\ x = a_2 \\ \alpha_{\tilde{a}}\left(\dfrac{a_3-x}{a_3-a_2}\right) & if\ a_2 < x \leq a_3 \\ 0 & otherwise \end{cases} \quad (1)$$

$$I_{\tilde{a}}(x) = \begin{cases} \dfrac{\left(a_2 - x + \theta_{\tilde{a}}(x-a_1)\right)}{(a_2-a_1)} & if\ a_1 \leq x \leq a_2 \\ \theta_{\tilde{a}} & if\ x = a_2 \\ \dfrac{(x - a_2 + \theta_{\tilde{a}}(a_3-x))}{(a_3-a_2)} & if\ a_2 < x \leq a_3 \\ 1 & otherwise \end{cases} \quad (2)$$

$$F_{\tilde{a}}(x) = \begin{cases} \dfrac{\left(a_2 - x + \beta_{\tilde{a}}(x-a_1)\right)}{(a_2-a_1)} & if\ a_1 \leq x \leq a_2 \\ \beta_{\tilde{a}} & if\ x = a_2 \\ \dfrac{\left(x - a_2 + \beta_{\tilde{a}}\,(a_3-x)\right)}{(a_3-a_2)} & if\ a_2 < x \leq a_3 \\ 1 & otherwise \end{cases} \quad (3)$$

where $\alpha_{\tilde{a}}, \theta_{\tilde{a}}$ and $\beta_{\tilde{a}}$ denote the maximum truth-membership degree, minimum indeterminacy-membership degree and minimum falsity-membership degree respectively. A single valued triangular neutrosophic number $\tilde{\boldsymbol{a}}$=$\langle(\boldsymbol{a_1}, \boldsymbol{a_2}, \boldsymbol{a_3}); \alpha_{\tilde{a}}, \theta_{\tilde{a}}, \beta_{\tilde{a}}\rangle$ may express an ill-defined quantity about $\boldsymbol{a}$,which is approximately equal to $\boldsymbol{a}$.

Definition 4. Let $\tilde{\boldsymbol{a}}=\langle(\boldsymbol{a_1},\boldsymbol{a_2},\boldsymbol{a_3});\alpha_{\tilde{a}},\theta_{\tilde{a}},\beta_{\tilde{a}}\rangle$ and $\tilde{\boldsymbol{b}}=\langle(\boldsymbol{b_1},\boldsymbol{b_2},\boldsymbol{b_3});\alpha_{\tilde{b}},\theta_{\tilde{b}},\beta_{\tilde{b}}\rangle$ be two single valued triangular neutrosophic numbers and $\gamma \neq 0$ be any real number. Then,

$$\tilde{\boldsymbol{a}}+\tilde{\boldsymbol{b}}=\langle(\boldsymbol{a_1}+\boldsymbol{b_1},\boldsymbol{a_2}+\boldsymbol{b_2},\boldsymbol{a_3}+\boldsymbol{b_3});\alpha_{\tilde{a}}\wedge\alpha_{\tilde{b}},\theta_{\tilde{a}}\vee\theta_{\tilde{b}},\beta_{\tilde{a}}\vee\beta_{\tilde{b}}\rangle$$

$$\tilde{\boldsymbol{a}}-\tilde{\boldsymbol{b}}=\langle(\boldsymbol{a_1}-\boldsymbol{b_3},\boldsymbol{a_2}-\boldsymbol{b_2},\boldsymbol{a_3}-\boldsymbol{b_1});\alpha_{\tilde{a}}\wedge\alpha_{\tilde{b}},\theta_{\tilde{a}}\vee\theta_{\tilde{b}},\beta_{\tilde{a}}\vee\beta_{\tilde{b}}\rangle$$

$$\tilde{\boldsymbol{a}}\tilde{\boldsymbol{b}}=\begin{cases}\langle(\boldsymbol{a_1}\boldsymbol{b_1},\boldsymbol{a_2}\boldsymbol{b_2},\boldsymbol{a_3}\boldsymbol{b_3});\alpha_{\tilde{a}}\wedge\alpha_{\tilde{b}},\theta_{\tilde{a}}\vee\theta_{\tilde{b}},\beta_{\tilde{a}}\vee\beta_{\tilde{b}}\rangle \\ \quad \boldsymbol{if}\ (\boldsymbol{a_3}>\boldsymbol{0},\boldsymbol{b_3}>\boldsymbol{0}) \\ \langle(\boldsymbol{a_1}\boldsymbol{b_3},\boldsymbol{a_2}\boldsymbol{b_2},\boldsymbol{a_3}\boldsymbol{b_1});\alpha_{\tilde{a}}\wedge\alpha_{\tilde{b}},\theta_{\tilde{a}}\vee\theta_{\tilde{b}},\beta_{\tilde{a}}\vee\beta_{\tilde{b}}\rangle \\ \quad \boldsymbol{if}\ (\boldsymbol{a_3}<\boldsymbol{0},\boldsymbol{b_3}>\boldsymbol{0}) \\ \langle(\boldsymbol{a_3}\boldsymbol{b_3},\boldsymbol{a_2}\boldsymbol{b_2},\boldsymbol{a_1}\boldsymbol{b_1});\alpha_{\tilde{a}}\wedge\alpha_{\tilde{b}},\theta_{\tilde{a}}\vee\theta_{\tilde{b}},\beta_{\tilde{a}}\vee\beta_{\tilde{b}}\rangle \\ \quad \boldsymbol{if}\ (\boldsymbol{a_3}<\boldsymbol{0},\boldsymbol{b_3}<\boldsymbol{0})\end{cases}$$

$$\frac{\tilde{\boldsymbol{a}}}{\tilde{\boldsymbol{b}}}=\begin{cases}\langle\left(\frac{\boldsymbol{a_1}}{\boldsymbol{b_3}},\frac{\boldsymbol{a_2}}{\boldsymbol{b_2}},\frac{\boldsymbol{a_3}}{\boldsymbol{b_1}}\right);\alpha_{\tilde{a}}\wedge\alpha_{\tilde{b}},\theta_{\tilde{a}}\vee\theta_{\tilde{b}},\beta_{\tilde{a}}\vee\beta_{\tilde{b}}\rangle & \boldsymbol{if}(\boldsymbol{a_3}>\boldsymbol{0},\boldsymbol{b_3}>\boldsymbol{0}) \\ \langle\left(\frac{\boldsymbol{a_3}}{\boldsymbol{b_3}},\frac{\boldsymbol{a_2}}{\boldsymbol{b_2}},\frac{\boldsymbol{a_1}}{\boldsymbol{b_1}}\right);\alpha_{\tilde{a}}\wedge\alpha_{\tilde{b}},\theta_{\tilde{a}}\vee\theta_{\tilde{b}},\beta_{\tilde{a}}\vee\beta_{\tilde{b}}\rangle & \boldsymbol{if}(\boldsymbol{a_3}<\boldsymbol{0},\boldsymbol{b_3}>\boldsymbol{0}) \\ \langle\left(\frac{\boldsymbol{a_3}}{\boldsymbol{b_1}},\frac{\boldsymbol{a_2}}{\boldsymbol{b_2}},\frac{\boldsymbol{a_1}}{\boldsymbol{b_3}}\right);\alpha_{\tilde{a}}\wedge\alpha_{\tilde{b}},\theta_{\tilde{a}}\vee\theta_{\tilde{b}},\beta_{\tilde{a}}\vee\beta_{\tilde{b}}\rangle & \boldsymbol{if}(\boldsymbol{a_3}<\boldsymbol{0},\boldsymbol{b_3}<\boldsymbol{0})\end{cases}$$

$$\gamma\tilde{\boldsymbol{a}}=\begin{cases}\langle(\gamma\boldsymbol{a_1},\gamma\boldsymbol{a_2},\gamma\boldsymbol{a_3});\alpha_{\tilde{a}},\theta_{\tilde{a}},\beta_{\tilde{a}}\rangle\ \boldsymbol{if}(\gamma>\boldsymbol{0}) \\ \langle(\gamma\boldsymbol{a_3},\gamma\boldsymbol{a_2},\gamma\boldsymbol{a_1});\alpha_{\tilde{a}},\theta_{\tilde{a}},\beta_{\tilde{a}}\rangle\ \boldsymbol{if}\ (\gamma<\boldsymbol{0})\end{cases}$$

$$\tilde{a}^{-1}=\langle\left(\frac{1}{a_3},\frac{1}{a_2},\frac{1}{a_1}\right);\alpha_{\tilde{a}},\theta_{\tilde{a}},\beta_{\tilde{a}}\rangle,\quad \text{where } (\tilde{a}\neq 0).$$

3 Critical Path Method in Neutrosophic Environment and the Proposed Algorithm

Project network is a set of activities that must be performed according to precedence constraints determining which activities must start after the completion of specified other activities. Let us define some terms used in drawing network diagram of CPM:

- Activity: It is any portion of a project that has a definite beginning and ending and may use some resources such as time, labor, material, equipment, etc.
- Event or Node: Beginning and ending points of activities denoted by circles are called nodes or events.
- Critical Path: Is the longest path in the network.

The problems of determining critical activities, events and paths are easy

ones in a network with deterministic (crisp) duration of activities and for this reason; in this section, we convert the neutrosophic CPM to its equivalent crisp model.

The CPM in neutrosophic environment takes the following form:

A network N= $\langle E, A, \tilde{T}\rangle$, being a project model, is given. E is asset of events (nodes) and A$\subset E\times E$ is a set of activities. $\tilde{T}$ is a triangular neutrosophic number and stand for activity duration.

To obtain crisp model of neutrosophic CPM we should use the following equations:

We defined a method to compare any two-single valued triangular neutrosophic numbers which is based on the score function and the accuracy function. Let $\tilde{a} = \langle (a_1, b_1, c_1), \alpha_{\tilde{a}}, \theta_{\tilde{a}}, \beta_{\tilde{a}} \rangle$ be a single valued triangular neutrosophic number, then

$$S(\tilde{a}) = \frac{1}{16}[a_1 + b_1 + c_1]\times\left(2 + \alpha_{\tilde{a}} - \theta_{\tilde{a}} - \beta_{\tilde{a}}\right) \quad (4)$$

and

$$A(\tilde{a}) = \frac{1}{16}[a_1 + b_1 + c_1]\times\left(2 + \alpha_{\tilde{a}} - \theta_{\tilde{a}} + \beta_{\tilde{a}}\right) \quad (5)$$

Is called the score and accuracy degrees of $\tilde{a}$, respectively. The neutrosophic CPM model can be represented by a crisp model using truth membership, indeterminacy membership, and falsity membership functions and the score and accuracy degrees of ã, using equations (1), (2), (3) and (4), (5) respectively.

Then the CPM with crisp activity times becomes:

A network N= $\langle E, A, T\rangle$, being a project model, is given. E is asset of events (nodes) and A$\subset E\times E$ is a set of activities. The set E=$\{1,2,\dots,n\}$ is labelled in such a way that the following condition holds: (i, j)$\in A$ and $i< j$. The activity times in the network are determined by $T_{ij.}$

Notations of CPM solution

T_i^e=Earliest occurrence time of predecessor event *i*,

T_i^l= Latest occurrence time of predecessor event *i*,

T_j^e=Earliest occurrence time of successor event *j*,

T_j^l= Latest occurrence time of successor event *j*,

T_{ij}^e/Start= Earliest start time of an activity *ij*,

T_{ij}^e/Finish=Earliest finish time of an activity *ij*,

T_{ij}^{l}/Start=Latest start time of an T_{i}^{l}activity ij,

T_{ij}^{l}/Finish= Latest finish time of an activity ij,

T_{ij} = Duration time of activity ij,

Earliest and Latest occurrence time of an event:

T_{j}^{e}=maximum $(T_{j}^{e} + T_{ij})$, calculate all T_{j}^{e} for jth event, select maximum value.

T_{i}^{l}=minimum $(T_{j}^{l} - T_{ij})$, calculate all T_{i}^{l} for ith event, select minimum value.

T_{ij}^{e}/Start=T_{i}^{e},

T_{ij}^{e}/Finish=$T_{i}^{e} + T_{ij}$,

T_{ij}^{l}/Finish=T_{j}^{l},

T_{ij}^{l}/Start=$T_{j}^{l} - T_{ij}$,

Critical path is the longest path in the network. At critical path, T_{i}^{e}=T_{i}^{l}, for all i.

Slack or Float is cushion available on event/ activity by which it can be delayed without affecting the project completion time.

Slack for ith event = $T_{i}^{l} - T_{i}^{e}$, for events on critical path, slack is zero.

From the previous steps we can conclude the proposed algorithm as follows:

Step1: To deal with uncertain, inconsistent and incomplete information about activity time, we considered activity time of CPM technique as triangular neutrosophic number.

Step 2: Calculate membership functions of each triangular neutrosophic number, using equation 1, 2 and 3.

Step 3: Obtain crisp model of neutrosophic CPM using equation (4) and (5) as we illustrated previously.

Step 4: Draw CPM network diagram.

Step 5: Determine floats and critical path, which is the longest path in network.

Step 6: Determine expected project completion time.

4 Illustrative Examples

To explain the proposed approach in a better way, we solved two numerical examples and steps of solution are determined clearly.

A. Numerical Example 1

An application deals with the realization of a road connection between two famous cities in Egypt namely Cairo and Zagazig. Linguistics terms such as "approximately between" and "around" can be properly represented by approximate reasoning of neutrosophic set theory. Here triangular neutrosophic numbers are used to describe the duration of each task of project. As a real time application of this model, the following example is considered. The project manager wishes to construct a possible route from Cairo (*s*) to Zagazig (*d*). Given a road map of Egypt on which the times taken between each pair of successive intersection are marked, to determine the critical path from source vertex (s) to the destination vertex (*d*). Activities and their neutrosophic durations are presented in table 1.

Table 1. Input data for neutrosophic cpm.

Activity	**Neutrosophic Activity Time(days)**	**Immediate predecessors**
A	About 2 days (1,2,3;0.8,0.5,0.3)	-
B	About 3 days (2,3,8;0.6,0.3,0.5)	-
C	About 3 days (1,3,10;0.9,0.7,0.6)	A
D	About 2 days (1,2,6;0.5,0.6,0.4)	B
E	About 5 days (2,5,11;0.8,0.6,0.7)	B
F	About 4 days (1,4,8;0.4,0.6,0.8)	C
G	About 5 days (3,5,20;0.8,0.3,0.2)	C
H	About 6 days (4,6,10;0.8,0.5,0.3)	D
I	About 7 days (5,7,15;0.3,0.5,0.4)	F ,E
J	About 5 days (3,5,7;0.8,0.5,0.7)	H, G

Step 1: Neutrosophic model of project take the following form:

N= $\langle E, A, \tilde{T} \rangle$, where E is asset of events (nodes) and A$\subset E \times E$ is a set of activities. $\tilde{T}$ is a triangular neutrosophic number and stand for activity time.

Step 2: Obtaining crisp model of problem by using equations (4) and (5). Activities and their crisp durations are presented in table 2.

Table 2. Input data for crisp cpm.

Activity	Activity Time(days)	Immediate predecessors
A	2	-
B	1	-
C	1	A
D	1	B
E	2	B
F	1	C
G	4	C
H	3	D
I	2	F,E
J	1	H,G

Step 3: Draw network diagram of CPM.

Network diagram of CPM using Microsoft Project 2010 presented in Fig.1.

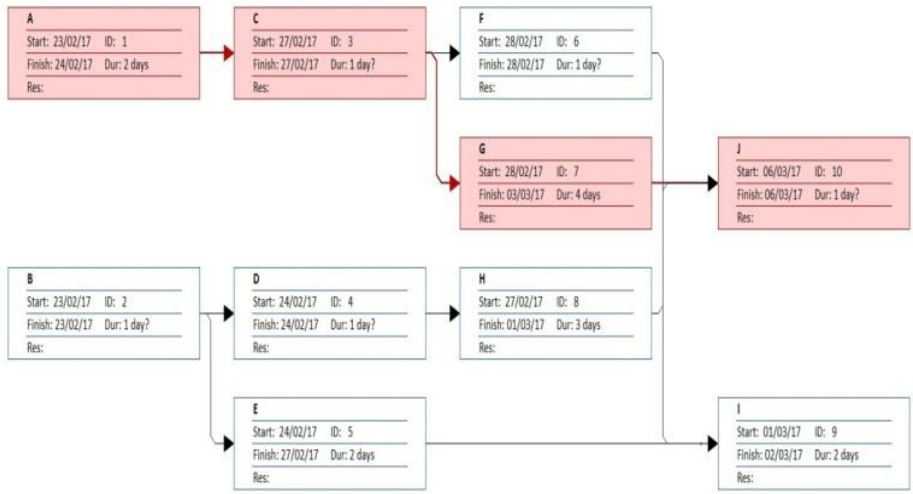

Fig. 1. Network of activities with critical path

Step 4: Determine critical path, which is the longest path in the network.

From Fig.1, we find that the critical path is A-C-G-J and is denoted by red line.

Step 5: Calculate project completion time.

The expected project completion time = $t_A + t_C + t_G + t_J$ = 8 days.

B. Numerical Example 2

Let us consider neutrosophic CPM and try to obtain crisp model from it. Since you are given the following data for a project.

Table 3. Input data for neutrosophic cpm.

Activity	**Neutrosophic Activity Time(days)**	**Immediate predecessors**
A	$\tilde{2}$	-
B	$\tilde{4}$	A
C	$\tilde{5}$	A
D	$\tilde{8}$	B
E	$\tilde{6}$	C
F	$\widetilde{10}$	D,E

Time in the previous table considered as a triangular neutrosophic numbers.

Let,

$\tilde{2} = \langle(0,2,4); 0.8,0.6,0.4\rangle, \tilde{8} = \langle(4,8,15);\ \ 0.2,0.3,0.5\rangle,$

$\tilde{4} = \langle(1,4,12); 0.2,0.5,0.6\rangle, \tilde{6} = \langle(2,6,18); 0.5,0.4,0.9\rangle,$

$\tilde{5} = \langle(1,5,10); 0.8,0.2,0.4\rangle, \widetilde{10} = \langle(2,10,22); 0.7,0.2,0.5\rangle.$

To obtain crisp values of each triangular neutrosophic number, we should calculate score function of each neutrosophic number using equation (4).

The expected time of each activity are presented in table 4.

Table 4. Input data for crisp cpm.

Activity	**Immediate Predecessors**	**Activity Time(days)**
A	-	1
B	A	1
C	A	2
D	B	2
E	C	2
F	D,E	4

After obtaining crisp values of activity time we can solve the critical path method easily, and determine critical path efficiently.

To draw network of activities with critical path we used Microsoft project program.

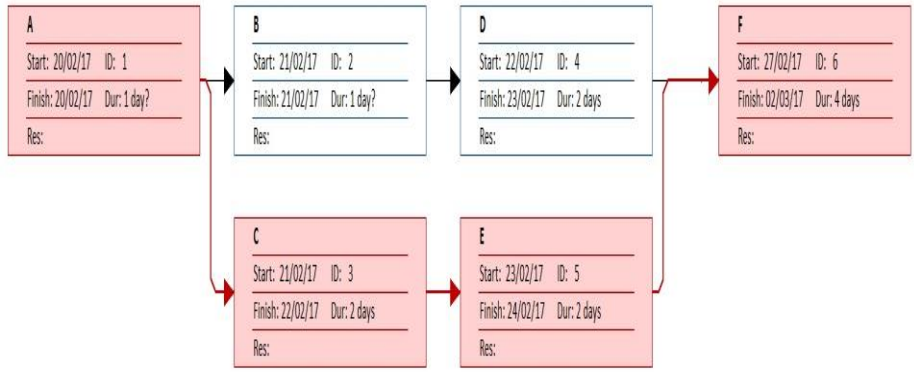

Fig. 2. Network of activities with critical path

From Fig.2, we find that the critical path is A-C-E-F and is denoted by red line.

The expected project completion time = $t_A + t_C + t_E + t_F$ = 9 days.

5 Conclusion

Neutrosophic set is a generalization of classical set, fuzzy set and intuitionistic fuzzy set because it not only considers the truth-membership and falsity- membership but also an indeterminacy function which is very obvious in real life situations. In this chapter, we have considered activity time of CPM as triangular neutrosophic numbers and we used score function to obtain crisp values of activity time. In future, the research will be extended to deal with different project management techniques.

Acknowledgment

The authors would like to thank anonymous referees for the constructive suggestions that improved both the quality and clarity of the chapter.

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XI

A Critical Path Problem in Neutrosophic Environment

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Abstract

The Critical Path Method (CPM) is one of several related techniques for planning and managing of complicated projects in real world applications. In many situations, the data obtained for decision makers are only approximate, which gives rise of neutrosophic critical path problem. In this chapter, the proposed method has been made to find the critical path in network diagram, whose activity time uncertain. The vague parameters in the network are represented by triangular neutrosophic numbers, instead of crisp numbers. At the end of chapter, an illustrative example is provided to validate the proposed approach**.**

Keywords

Neutrosophic Sets; Project Management; CPM; Score and Accuracy Functions.

1 Introduction

Project management is concerned with selecting, planning, execution and control of projects in order to meet or exceed stakeholders' need or expectation from project. Two techniques of project management, namely Critical Path Method (CPM) and Program Evaluation and Review Technique (PERT) where

developed in 1950s. [1] The successful implementation of CPM requires clear determined time duration for each activity.

Steps involved in CPM include [2]:

- Develop Work Breakdown Structure of a project, estimate the resources needed and establish precedence relationship among activities.
- Translate the activities into network.
- Carry out network computation and prepare schedule of the activities.

In CPM, the main problem is wrongly calculated activity durations, of large projects that have many activities. The planned value of activity duration time may change under certain circumstances and may not be presented in a precise manner due to the error of the measuring technique or instruments etc. It has been obvious that neutrosophic set theory is more appropriate to model uncertainty that is associated with parameters such as activity duration time and resource availability in CPM.

This chapter is organized as follows: In section 2, the basic concepts neutrosophic sets are briefly reviewed. In section 3, the mathematical model of neutrosophic CPM and the proposed algorithm is presented. In section 4, a numerical example is illustrated. Finally, section 5 concludes the chapter with future work.

2 Preliminaries

In this section, the basic definitions involving neutrosophic set, single valued neutrosophic sets, triangular neutrosophic numbers and operations on triangular neutrosophic numbers are outlined.

Definition 1. [3, 5-7] Let X be a space of points (objects) and $x \in X$. A neutrosophic set A in X is defined by a truth-membership function $T_A(x)$, an indeterminacy-membership function $I_A(x)$ and a falsity-membership function $F_A(x)$. $T_A(x)$, $I_A(x)$ and $F_A(x)$ are real standard or real nonstandard subsets of $]^-0, 1^+[$. That is $T_A(x): X \to]^-0, 1^+[$, $I_A(x): X \to]^-0, 1^+[$ and $F_A(x): X \to]^-0, 1^+[$. There is no restriction on the sum of $T_A(x)$, $I_A(x)$ and $F_A(x)$, so $0^- \leq \sup T_A(x) + \sup I_A(x) + \sup F_A(x) \leq 3^+$.

Definition 2. [3, 8] Let X be a universe of discourse. A single valued neutrosophic set A over X is an object having the form $A = \{\langle x, \boldsymbol{T_A}(x), \boldsymbol{I_A}(x), \boldsymbol{F_A}(x)\rangle : x \in X\}$, where $\boldsymbol{T_A}(x): X \to [0,1]$, $\boldsymbol{I_A}(x): X \to [0,1]$ and $\boldsymbol{F_A}(x): X \to [0,1]$ with $0 \leq \boldsymbol{T_A}(x) + \boldsymbol{I_A}(x) + \boldsymbol{F_A}(x) \leq 3$ for all $x \in X$. The intervals $\boldsymbol{T_A}(x)$, $\boldsymbol{I_A}(x)$ and $\boldsymbol{F_A}(x)$ denote the

truth-membership degree, the indeterminacy-membership degree and the falsity membership degree of x to A, respectively. For convenience, a SVN number is denoted by A= (a, b, c), where a, b, $c\in$ [0, 1] and a+b+$c\leq$3.

Definition 3. [4, 5] Let $\alpha_{\tilde{a}}, \theta_{\tilde{a}}, \beta_{\tilde{a}} \in [\mathbf{0}, \mathbf{1}]$ and $\boldsymbol{a_1}, \boldsymbol{a_2}, \boldsymbol{a_3} \in \boldsymbol{R}$ such that $\boldsymbol{a_1} \leq \boldsymbol{a_2} \leq \boldsymbol{a_3}$. Then a single valued triangular neutrosophic number, $\tilde{\boldsymbol{a}}=\langle(\boldsymbol{a_1}, \boldsymbol{a_2}, \boldsymbol{a_3}); \alpha_{\tilde{a}}, \theta_{\tilde{a}}, \beta_{\tilde{a}}\rangle$ is a special neutrosophic set on the real line set $\boldsymbol{R}$, whose truth-membership, indeterminacy-membership, and falsity-membership functions are given as follows:

$$T_{\tilde{a}}(x) = \begin{cases} \alpha_{\tilde{a}}\left(\dfrac{x-a_1}{a_2-a_1}\right) & if\ a_1 \leq x \leq a_2 \\ \alpha_{\tilde{a}} & if\ x = a_2 \\ \alpha_{\tilde{a}}\left(\dfrac{a_3-x}{a_3-a_2}\right) & if\ a_2 < x \leq a_3 \\ 0 & otherwise \end{cases} \quad (1)$$

$$I_{\tilde{a}}(x) = \begin{cases} \dfrac{\left(a_2 - x + \theta_{\tilde{a}}(x-a_1)\right)}{(a_2-a_1)} & if\ a_1 \leq x \leq a_2 \\ \theta_{\tilde{a}} & if\ x = a_2 \\ \dfrac{(x-a_2+\theta_{\tilde{a}}(a_3-x))}{(a_3-a_2)} & if\ a_2 < x \leq a_3 \\ 1 & otherwise \end{cases} \quad (2)$$

$$F_{\tilde{a}}(x) = \begin{cases} \dfrac{\left(a_2 - x + \beta_{\tilde{a}}(x-a_1)\right)}{(a_2-a_1)} & if\ a_1 \leq x \leq a_2 \\ \beta_{\tilde{a}} & if\ x = a_2 \\ \dfrac{\left(x-a_2+\beta_{\tilde{a}}(a_3-x)\right)}{(a_3-a_2)} & if\ a_2 < x \leq a_3 \\ 1 & otherwise \end{cases} \quad (3)$$

where $\alpha_{\tilde{a}}, \theta_{\tilde{a}}$ and $\beta_{\tilde{a}}$ denote the maximum truth-membership degree, minimum indeterminacy-membership degree and minimum falsity-membership degree respectively. A single valued triangular neutrosophic number $\tilde{\boldsymbol{a}}=\langle(\boldsymbol{a_1}, \boldsymbol{a_2}, \boldsymbol{a_3}); \alpha_{\tilde{a}}, \theta_{\tilde{a}}, \beta_{\tilde{a}}\rangle$ may express an ill-defined quantity about $\boldsymbol{a}$,which is approximately equal to $\boldsymbol{a}$.

Definition 4. Let $\tilde{\boldsymbol{a}}=\langle(\boldsymbol{a_1},\boldsymbol{a_2},\boldsymbol{a_3});\alpha_{\tilde{a}},\theta_{\tilde{a}},\beta_{\tilde{a}}\rangle$ and $\tilde{\boldsymbol{b}}=\langle(\boldsymbol{b_1},\boldsymbol{b_2},\boldsymbol{b_3});\alpha_{\tilde{b}},\theta_{\tilde{b}},\beta_{\tilde{b}}\rangle$ be two single valued triangular neutrosophic numbers and $\gamma \neq 0$ be any real number. Then,

$$\tilde{\boldsymbol{a}}+\tilde{\boldsymbol{b}}=\langle(\boldsymbol{a_1}+\boldsymbol{b_1},\boldsymbol{a_2}+\boldsymbol{b_2},\boldsymbol{a_3}+\boldsymbol{b_3});\alpha_{\tilde{a}}\wedge\alpha_{\tilde{b}},\theta_{\tilde{a}}\vee\theta_{\tilde{b}},\beta_{\tilde{a}}\vee\beta_{\tilde{b}}\rangle$$

$$\tilde{\boldsymbol{a}}-\tilde{\boldsymbol{b}}=\langle(\boldsymbol{a_1}-\boldsymbol{b_3},\boldsymbol{a_2}-\boldsymbol{b_2},\boldsymbol{a_3}-\boldsymbol{b_1});\alpha_{\tilde{a}}\wedge\alpha_{\tilde{b}},\theta_{\tilde{a}}\vee\theta_{\tilde{b}},\beta_{\tilde{a}}\vee\beta_{\tilde{b}}\rangle$$

$$\tilde{\boldsymbol{a}}\tilde{\boldsymbol{b}}=\begin{cases}\langle(\boldsymbol{a_1}\boldsymbol{b_1},\boldsymbol{a_2}\boldsymbol{b_2},\boldsymbol{a_3}\boldsymbol{b_3});\alpha_{\tilde{a}}\wedge\alpha_{\tilde{b}},\theta_{\tilde{a}}\vee\theta_{\tilde{b}},\beta_{\tilde{a}}\vee\beta_{\tilde{b}}\rangle \\ \quad \boldsymbol{if}\ (\boldsymbol{a_3}>\boldsymbol{0},\boldsymbol{b_3}>\boldsymbol{0}) \\ \langle(\boldsymbol{a_1}\boldsymbol{b_3},\boldsymbol{a_2}\boldsymbol{b_2},\boldsymbol{a_3}\boldsymbol{b_1});\alpha_{\tilde{a}}\wedge\alpha_{\tilde{b}},\theta_{\tilde{a}}\vee\theta_{\tilde{b}},\beta_{\tilde{a}}\vee\beta_{\tilde{b}}\rangle \\ \quad \boldsymbol{if}\ (\boldsymbol{a_3}<\boldsymbol{0},\boldsymbol{b_3}>\boldsymbol{0}) \\ \langle(\boldsymbol{a_3}\boldsymbol{b_3},\boldsymbol{a_2}\boldsymbol{b_2},\boldsymbol{a_1}\boldsymbol{b_1});\alpha_{\tilde{a}}\wedge\alpha_{\tilde{b}},\theta_{\tilde{a}}\vee\theta_{\tilde{b}},\beta_{\tilde{a}}\vee\beta_{\tilde{b}}\rangle \\ \quad \boldsymbol{if}\ (\boldsymbol{a_3}<\boldsymbol{0},\boldsymbol{b_3}<\boldsymbol{0})\end{cases}$$

$$\frac{\tilde{\boldsymbol{a}}}{\tilde{\boldsymbol{b}}}=\begin{cases}\langle\left(\frac{\boldsymbol{a_1}}{\boldsymbol{b_3}},\frac{\boldsymbol{a_2}}{\boldsymbol{b_2}},\frac{\boldsymbol{a_3}}{\boldsymbol{b_1}}\right);\alpha_{\tilde{a}}\wedge\alpha_{\tilde{b}},\theta_{\tilde{a}}\vee\theta_{\tilde{b}},\beta_{\tilde{a}}\vee\beta_{\tilde{b}}\rangle\ \ \boldsymbol{if}(\boldsymbol{a_3}>\boldsymbol{0},\boldsymbol{b_3}>\boldsymbol{0}) \\ \langle\left(\frac{\boldsymbol{a_3}}{\boldsymbol{b_3}},\frac{\boldsymbol{a_2}}{\boldsymbol{b_2}},\frac{\boldsymbol{a_1}}{\boldsymbol{b_1}}\right);\alpha_{\tilde{a}}\wedge\alpha_{\tilde{b}},\theta_{\tilde{a}}\vee\theta_{\tilde{b}},\beta_{\tilde{a}}\vee\beta_{\tilde{b}}\rangle\ \ \boldsymbol{if}(\boldsymbol{a_3}<\boldsymbol{0},\boldsymbol{b_3}>\boldsymbol{0}) \\ \langle\left(\frac{\boldsymbol{a_3}}{\boldsymbol{b_1}},\frac{\boldsymbol{a_2}}{\boldsymbol{b_2}},\frac{\boldsymbol{a_1}}{\boldsymbol{b_3}}\right);\alpha_{\tilde{a}}\wedge\alpha_{\tilde{b}},\theta_{\tilde{a}}\vee\theta_{\tilde{b}},\beta_{\tilde{a}}\vee\beta_{\tilde{b}}\rangle\ \ \boldsymbol{if}(\boldsymbol{a_3}<\boldsymbol{0},\boldsymbol{b_3}<\boldsymbol{0})\end{cases}$$

$$\gamma\tilde{\boldsymbol{a}}=\begin{cases}\langle(\gamma\boldsymbol{a_1},\gamma\boldsymbol{a_2},\gamma\boldsymbol{a_3});\alpha_{\tilde{a}},\theta_{\tilde{a}},\beta_{\tilde{a}}\rangle\ \ \boldsymbol{if}(\gamma>\boldsymbol{0}) \\ \langle(\gamma\boldsymbol{a_3},\gamma\boldsymbol{a_2},\gamma\boldsymbol{a_1});\alpha_{\tilde{a}},\theta_{\tilde{a}},\beta_{\tilde{a}}\rangle\ \ \boldsymbol{if}\ (\gamma<\boldsymbol{0})\end{cases}$$

$$\tilde{a}^{-1}=\langle\left(\frac{1}{a_3},\frac{1}{a_2},\frac{1}{a_1}\right);\alpha_{\tilde{a}},\theta_{\tilde{a}},\beta_{\tilde{a}}\rangle,\quad \text{where } (\tilde{a}\neq 0).$$

3 Critical Path Method in Neutrosophic Environment and the Proposed Algorithm

Project network is a set of activities that must be performed according to precedence constraints determining which activities must start after the completion of specified other activities. Let us define some terms used in drawing network diagram of CPM:

- Activity: It is any portion of a project that has a definite beginning and ending and may use some resources such as time, labor, material, equipment, etc.
- Event or Node: Beginning and ending points of activities denoted by circles are called nodes or events.
- Critical Path: Is the longest path in the network.

The CPM in neutrosophic environment takes the following form:
A network N= $\langle E, A, \tilde{T}\rangle$, being a project model, is given. E is asset of events

(nodes) and A$\subset E\times E$ is a set of activities. $\tilde{T}$ is a triangular neutrosophic number and stand for activity duration.

To obtain crisp model of neutrosophic CPM we should use the following equations:

We defined a method to compare any two single valued triangular neutrosophic numbers which is based on the score function and the accuracy function. Let $\tilde{a} = \langle (a_1, b_1, c_1), \alpha_{\tilde{a}}, \theta_{\tilde{a}}, \beta_{\tilde{a}} \rangle$ be a single valued triangular neutrosophic number, then

$$S(\tilde{a}) = \frac{1}{16}[a_1 + b_1 + c_1]\times\left(2 + \alpha_{\tilde{a}}\text{-}\theta_{\tilde{a}}\text{-}\beta_{\tilde{a}}\right) \quad (4)$$

and

$$A(\tilde{a}) = \frac{1}{16}[a_1 + b_1 + c_1]\times\left(2 + \alpha_{\tilde{a}}\text{-}\theta_{\tilde{a}} + \beta_{\tilde{a}}\right) \quad (5)$$

It is called the score and accuracy degrees of $\tilde{a}$, respectively. The neutrosophic CPM model can be represented by a crisp model using truth membership, indeterminacy membership, and falsity membership functions and the score and accuracy degrees of ã, using equations (1), (2), (3) and (4), (5) respectively.

Notations of CPM solution:

$\tilde{T}_i^e$=Earliest occurrence time of predecessor event *i*,

$\tilde{T}_i^l$= Latest occurrence time of predecessor event *i*,

$\tilde{T}_j^e$=Earliest occurrence time of successor event *j*,

$\tilde{T}_j^l$= Latest occurrence time of successor event *j*,

$\tilde{T}_{ij}^e$/Start= Earliest start time of an activity *ij*,

$\tilde{T}_{ij}^e$/Finish=Earliest finish time of an activity *ij*,

$\tilde{T}_{ij}^l$/Start=Latest start time of an T_i^lactivity *ij*,

$\tilde{T}_{ij}^l$/Finish= Latest finish time of an activity *ij*,

$\tilde{T}_{ij}$ = Duration time of activity *ij*,

Earliest and Latest occurrence time of an event:

$\tilde{T}_j^e$=maximum $(\tilde{T}_j^e + \tilde{T}_{ij})$, calculate all $\tilde{T}_j^e$ for jth event, select maximum value.

$\tilde{T}_i^l$=minimum $(\tilde{T}_j^l - \tilde{T}_{ij})$, calculate all T_i^l for ith event, select minimum value.

$\tilde{T}_{ij}^e$/Start=$\tilde{T}_i^e$,

$\tilde{T}_{ij}^e$/Finish=$\tilde{T}_i^e + \tilde{T}_{ij}$,

$\tilde{T}_{ij}^l$/Finish=$\tilde{T}_j^l$,

$\tilde{T}_{ij}^l$/Start=$\tilde{T}_j^l - \tilde{T}_{ij}$,

Critical path is the longest path in the network. At critical path, $\tilde{T}_i^e$=$\tilde{T}_i^l$, for all *i*, and don't care of the value of α,θ,β.

Slack or Float is cushion available on event/ activity by which it can be delayed without affecting the project completion time.

Slack for ith event =$\tilde{T}_i^l - \tilde{T}_i^e$, for events on critical path, slack is zero.

From the previous steps we can conclude the proposed algorithm as follows:

Step1: To deal with uncertain, inconsistent and incomplete information about activity time, we considered activity time of CPM technique as triangular neutrosophic number.

Step 2: Draw CPM network diagram.

Step 3: Determine floats and critical path, which is the longest path in network.

Step 4: Determine expected project completion time.

4 Illustrative Examples

To explain the proposed approach in a better way, we solved numerical example and steps of solution are determined clearly.

1.1. Numerical Example 1

You are given the following data for a project:

Table 1. Input data for neutrosophic cpm.

Activity	Immediate Predecessors	Time (days)
A (1-2)	------	$\tilde{3} = \langle(2,3,4); 0.6,0.3,0.1\rangle$
B (1-3)	------	$\tilde{5} = \langle(4,5,6); 0.8,0.2,0.4\rangle$
C (2-4)	A	$\tilde{4} = \langle(1,4,8); 0.8,0.6,0.4\rangle$
D (3-4)	B	$\tilde{6} = \langle(2,6,8); 0.6,0.4,0.2\rangle$
E (4-5)	C,D	$\tilde{8} = \langle(6,8,10); 0.6,0.4,0.4\rangle$

Step 1: Neutrosophic model of project take the following form:

N= $\langle E, A, \tilde{T} \rangle$, where E is asset of events (nodes) and A$\subset E \times E$ is a set of activities, $\tilde{T}$ is a triangular neutrosophic number and stand for activity time.

Step 2: Draw network diagram of CPM.

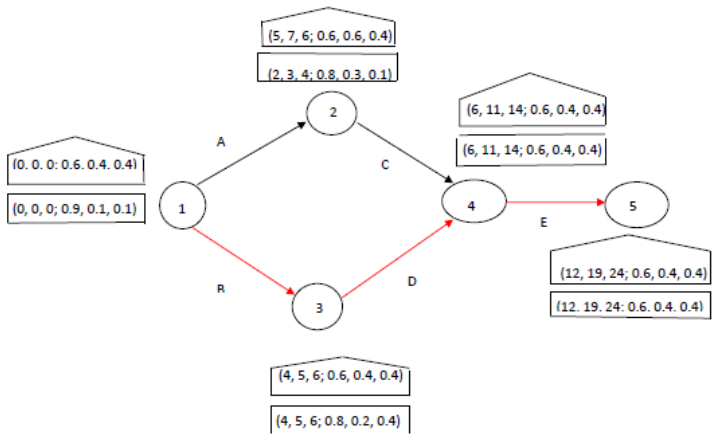
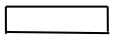
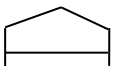


Fig.1. Network diagram of CPM

 ⇒ Determine earliest start/finish of each activity.

 ⇒ Determine latest start/finish of each activity.

Step 3: Determine critical path, which is the longest path in the network.

From Fig.1, we find that the critical path is B-D-E and is denoted by red line.

Step 4: Calculate project completion time.

The neutrosophic time of project completion = (12, 19, 24; 0.6, 0.4, 0.4)$t_A + t_C + t_G + t_J$ days.

To determine crisp value of project completion time we will use Eq.4, then the expected time of project completion in deterministic environment = 12 days.

5 Conclusion

Neutrosophic set is a generalization of classical set, fuzzy set and intuitionistic fuzzy set because it not only considers the truth-membership and falsity- membership but also an indeterminacy function which is very obvious in

real life situations. In this chapter, we have considered activity time of CPM as triangular neutrosophic numbers. In future, the research will be extended to deal with different project management techniques.

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XII

Integrated Framework of Optimization Technique and Information Theory Measures for Modeling Neutrosophic Variables

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Abstract

Uncertainty and indeterminacy are two major problems in data analysis these days. Neutrosophy is a generalization of the fuzzy theory. Neutrosophic system is based on indeterminism and falsity of concepts in addition to truth degrees. Any neutrosophy variable or concept is defined by membership, indeterminacy and non-membership functions. Finding efficient and accurate definition for neutrosophic variables is a challenging process. This chapter presents a framework of Ant Colony Optimization and entropy theory to define a neutrosophic variable from concrete data. Ant Colony Optimization is an efficient search algorithm presented to define parameters of membership, indeterminacy and non-membership functions. The integrated framework of information theory measures and Ant Colony Optimization is proposed. Experimental results contain graphical representation of the membership, indeterminacy and non-membership functions for the temperature variable of the forest fires data set. The graphs demonstrate the effectiveness of the proposed framework.

Keywords

Neutrosophic set, Ant Colony Optimization, Information Theory Measures, Entropy function.

1 Introduction

These days, Indeterminacy is the key idea of the information in reality issues. This term eludes to the obscure some portion of the information representation. The fuzzy logic [1], [2], [3], serves the piece of information participation degree. Thus, the indeterminacy and non-participation ideas of the information ought to be fittingly characterized and served. The neutrosophic [4], [16] theory characterizes the informational index in mix with their membership, indeterminacy and non-membership degrees. Thus, the decisions could be practically figured out from this well-defined information.

Smarandache in [5], [13], [14], and Salama et al. in [4], [7], [8], [9], [10] [11], [12], [16] present the mathematical base of neutrosophic system and principles of neutrosophic data. Neutrosophy creates the main basics for a new mathematics field through adding indeterminacy concept to traditional and fuzzy theories[1], [2], [3], [15].

Handling neutrosophic system is a new, moving and appealing field for scientists. In literature, neutrosophic toolbox implementation using object oriented programming operations and formulation is introduced in [18]. Moreover, a data warehouse utilizing neutrosophic methodologies and sets is applied in [17]. Also, the problem of optimizing membership functions using Particle Swarm Optimization was introduced in [24]. This same mechanism could be generalized to model neutrosophic variable.

The neutrosophic framework depends actually on the factors or variables as basics. The neutrosophic variable definition is without a doubt the base in building a precise and productive framework. The neutrosophic variable is made out of a tuple of value, membership, indeterminacy and non-membership. Pronouncing the elements of participation, indeterminacy and non-enrolment and map those to the variable values would be an attainable arrangement or solution for neutrosophic variable formulation.

Finding the subsets boundary points of membership and non-membership functions within a variable data would be an interesting optimization problem. Ant Colony Optimization (ACO) [19], [20] is a meta-heuristic optimization and search procedure [22] inspired by ants lifestyle in searching for food. ACO initializes a population of ants in the search space traversing for their food according to some probabilistic transition rule. Ants follow each other basing on rode pheromone level and ant desirability to go through a specific path. The main issue is finding suitable heuristic desirability which should be based on the information conveyed from the variable itself. Information theory measures [6], [20], [21], [23] collect information from concrete data. The entropy definition is

the measure of information conveyed in a variable. Whereas, the mutual information is the measure of data inside a crossing point between two nearby subsets of a variable. These definitions may help in finding limits of a membership function of neutrosophic variable subsets depending on the probability distribution of the data as the heuristic desirability of ants.

In a similar philosophy, the non- membership of a neutrosophic variable might be characterized utilizing the entropy and mutual information basing on the data probability distribution complement. Taking the upsides of the neutrosophic set definition; the indeterminacy capacity could be characterized from the membership and non-membership capacities.

This chapter exhibits an incorporated hybrid search model amongst ACO and information theory measures to demonstrate a neutrosophic variable. The rest of this chapter is organized as follows. Section 2 shows the hypotheses and algorithms. Section 3 announces the proposed integrated framework. Section 4 talks about the exploratory outcomes of applying the framework on a general variable and demonstrating the membership, indeterminacy and non-membership capacities. Conclusion and future work is displayed in section 5.

2 Theory Overview

2.1 Parameters of a neutrosophic variable

In the neutrosophy theory[5][13][14], every concept is determined by rates of truth $\mu_A(x)$, indeterminacy $\sigma_A(x)$, and negation $\nu_A(x)$ in various partitions. Neutrosophy is a generalization of the fuzzy hypothesis [1], [2], [3] and an extension of the regular set. Neutrosophic is connected to concepts identified with indeterminacy. Neutrosophic data is defined by three main concepts to manage uncertainty. These concepts are joined together in the triple:

$$A = \langle \mu_A(x), \sigma_A(x), \nu_A(x) \rangle \quad (1)$$

where

$\mu_A(x)$ is the membership degree,

$\sigma_A(x)$ is the indeterminacy degree,

$\nu_A(x)$ is the falsity degree.

These three terms form the fundamental concepts and they are independent and explicitly quantified. In neutrosophic set, each value $x \in X$ in set A defined by Eq. 1 is constrained by the following conditions:

$$0^- \leq \mu_A(x), \sigma_A(x), \nu_A(x) \leq 1^+ \quad (2)$$

$$0^- \leq \mu_A(x) + \sigma_A(x) + \nu_A(x) \leq 3^+ \quad (3)$$

Whereas, Neutrosophic intuitionistic set of type is subjected to the following:

$$0^- \leq \mu_A(x), \sigma_A(x), \nu_A(x) \leq 1^+ \quad (4)$$

$$\mu_A(x) \wedge \sigma_A(x) \wedge \nu_A(x) \leq 0.5 \quad (5)$$

$$0^- \leq \mu_A(x) + \sigma_A(x) + \nu_A(x) \leq 3^+ \quad (6)$$

Neutrosophic intuitionistic set of type 2 [5] is obliged by to the following conditions:

$$0.5 \leq \mu_A(x), \sigma_A(x), \nu_A(x) \quad (7)$$

$$\mu_A(x) \wedge \sigma_A(x) \leq 0.5, \ \mu_A(x) \wedge \nu_A(x) \leq 0.5, \ \sigma_A(x) \wedge \nu_A(x) \leq 0.5 \quad (8)$$

$$0^- \leq \mu_A(x) + \sigma_A(x) + \nu_A(x) \leq 2^+ \quad (9)$$

2.2 Ant Colony Optimization (ACO)

The ACO [19], [20]is an efficient search algorithm used to find feasible solutions for complex and high dimension problems. The intelligence of the ACO is based on a population of ants traversing the search workspace for their food. Each ant follows a specific path depending on information left previously from other ants. This information is characterized by the probabilistic transition rule Eq. 10.

$$p_j^m(t) = \frac{[\eta_j] \times [\tau_{ij}(t)]}{\sum_{i \in I_m} [\eta_i] \times [\tau_{ij}(t)]} \quad (10)$$

where:

η_j is the heuristic desirability of choosing node j and

τ_{ij} is the amount of virtual pheromone on edge (i, j)

The pheromone level guides the ant through its journey. This guide is a hint of the significance level of a node (exhibited by the ants went to the nodes some time recently). The pheromone level is updated by the algorithm using the fitness function.

$$\tau_{ij}(t+1) = (1-\rho).\tau_{ij}(t) + \Delta\tau_{ij}(\mathrm{t}) \quad (11)$$

where 0< ρ <1 is a decay constant used to estimate the evaporation of the pheromone from the edges. $\Delta\tau_{ij}(t)$ is the amount of pheromone deposited by the ant.

The heuristic desirability η_j describes the association between a node j and the problem solution or the fitness function of the search. If a node has a heuristic value for a certain path then the ACO will use this node in the solution of the problem. The algorithm of ACO is illustrated in figure 1.

$$\eta_j = objective\ function \tag{12}$$

ACO Algorithm

Input :pd, N

%%%% pd number of decision variables in ant, N iterations, Present position (ant) in the search universe X_{id} , ρ evaporation rate, %%%%%%%

Output: Best_Solution

1: Initianlize_Node_Graph();

2: Initialize_Phermoni_Node();

3: While (num_of_Iterations>0) do

4: for each Ant

5: η_j ***← objective function of the search space***

6: TRANSITION_RULE[j]= $p_j^m(t) = \frac{[\eta_j]\times[\tau_{ij}(t)]}{\sum_{i\in I_m}[\eta_i]\times[\tau_{ij}(t)]}$

7: Select node with the highest $p_j^m(t)$

8: Update Pheromone level $\tau_{ij}(t+1) = (1-\rho).\tau_{ij}(t) + \Delta\tau_{ij}(t)$

9: num_of_Iterations--;

10: end While

11:Best_sol ← solution with best η_j

12: output(Best_sol)

Fig. 1: Pseudo code of ant colony optimization Algorithm

2.3 Entropy and Mutual Information

Information theory measures [6], [20][23] collect information from raw data. The entropy of a random variable is a function which characterizes the

unexpected events of a random variable. Consider a random variable X expressing the number on a roulette wheel or the number on a fair 6-sided die.

$$H(X) = \sum_{x \in X} -P(x) \log P(x) \quad (13)$$

Joint entropy is the entropy of a joint probability distribution, or a multi-valued random variable. For example, consider the joint entropy of a distribution of mankind (X) defined by a characteristic (Y) like age or race or health status of a disease.

$$I(X;\ Y) = \sum_{x,y} p(x,y) \log \frac{p(x,y)}{p(x)p(y)} \quad (14)$$

3 The Proposed FrameWork

An Integrated hybrid model of ACO and information theory measures (entropy and mutual information) as the objective function is presented. The ACO[19][20] is a heuristic searching algorithm used to locate the ideal segments of the membership and non-membership functions of a neutrosophic variable. The indeterminacy function is calculated by the membership and non-membership functions basing on the definitions of neutrosophic set illustrated in section 2. The objective function is the amount of information conveyed from various partitions in the workspace. Therefore, the total entropy [21] is used as the objective function on the variables workspace. Total entropy calculates amount of information of various partitions and intersections between these partitions. Best points in declaring the membership function are the boundaries of the partitions. The ants are designed to form the membership and non-membership partitions as illustrated in figure 2. A typical triangle membership function would take the shape of figure 2.

The triangle function of a variable partition is represented by parameters (L, (L+U)/2, U). Finding best values of L and U for all partitions would optimize the membership (non-membership) function definition. Figure 3 give a view of the ant with n partitions for each fuzzy variable.

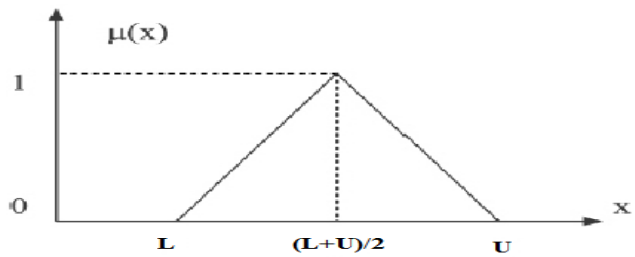


Fig. 2: corresponding to triangle fuzzy membership and its boundary parameters

Individual	L_1	U_1	L_2	U_2	……	L_n	U_n

Fig. 3: Individual in ACO for Triangle function

One of the main difficulties in designing optimization problem using ACO is finding the heuristic desirability which formulates the transition rule. The amount of information deposited by neutrosophic variable inspires the ACO to calculate the transition rule and find parameters of membership, indeterminism and non-membership declarations. The membership function subsets are declared by ant parameters in figure 2. The histogram of a variable shows the data distribution of the different values. Therefore, the set of parameters are mapped to the histogram of a given variable data (Fig. 4).

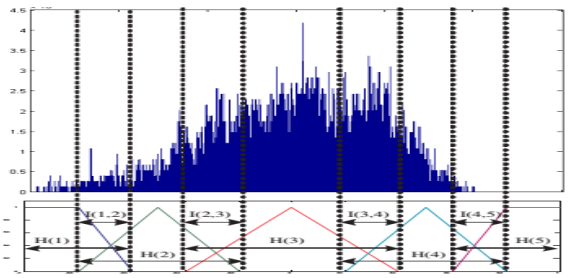


Fig. 4: Fuzzy discretizing of the histogram into n joint subsets and m-1 intersections

The objective function is set as the total entropy of partitions[23]. By enhancing partition's parameters to optimize the total entropy of the histogram subsets, the optimal membership design of the variable is found.

To model (n) membership functions, variable histogram is partitioned into n overlapped subsets that produce n-1 intersections. Every joint partition corresponds to joint entropy and each overlap is modelled by mutual information. Eq.15 shows the total entropy which is assigned to the heuristic desirability of ants.

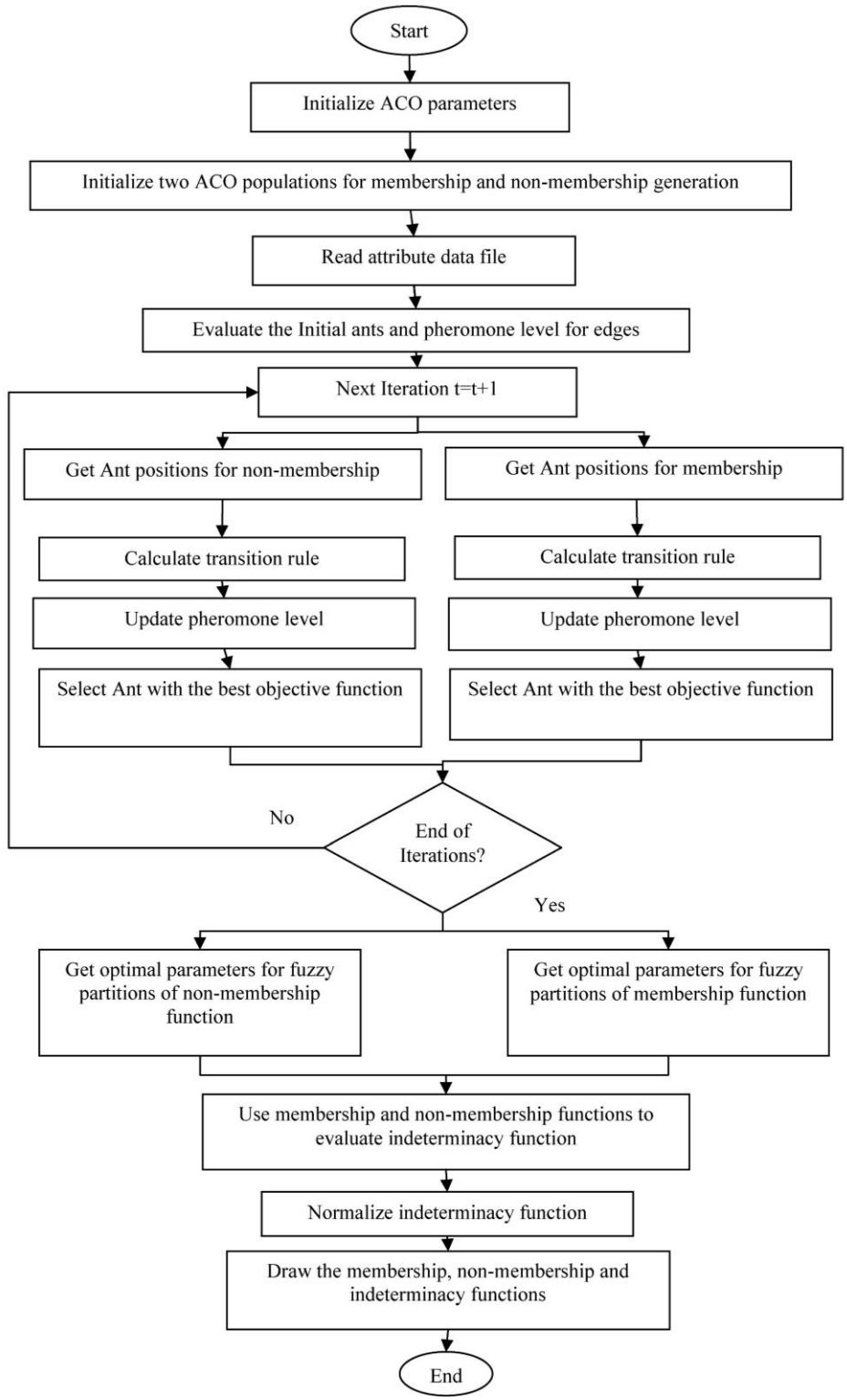
$$\eta_j = H = \sum_{i=1}^{n} H(i) - \sum_{j=1}^{n-1} I(j, j+1) \quad (15)$$

where n is the number of partitions or subsets in the fuzzy variable,

H is the total entropy,

$H(i)$ is the entropy of subset i,

I is the mutual information between to intersecting partitions (i, j).


Start
Initialize ACO parameters
Initialize two ACO populations for membership and non-membership generation
Read attribute data file
Evaluate the Initial ants and pheromone level for edges
Next Iteration t=t+1
Get Ant positions for non-membership
Get Ant positions for membership
Calculate transition rule
Calculate transition rule
Update pheromone level
Update pheromone level
Select Ant with the best objective function
Select Ant with the best objective function
No
End of Iterations?
Yes
Get optimal parameters for fuzzy partitions of non-membership function
Get optimal parameters for fuzzy partitions of membership function
Use membership and non-membership functions to evaluate indeterminacy function
Normalize indeterminacy function
Draw the membership, non-membership and indeterminacy functions
End

Input :pd, N, variable_datafile
%%%% pd number of decision variables in particle, N ieteration, Present position in the search universe X_{id} , ρ is the decay rate of phermone. %%%%%%%
Output: membership, non-membership and indeterminacy function, conversion rate.
1: X←Initianlize_Ants(); % Each ant is composed of **pd** decision variables for fuzzy partitions
2:Att←Read_data(variable_datafile)
3:Objective_mem_ ← Evaluate _ Objective_of_Particles (X, P(Att)); % According to entropy and Mutual information
4: Objective_non_mem ← Evaluate _ Objective_of_Particles (X, 1-P(Att)); % According to entropy and Mutual information

5: **While** (num_of_Iterations<Max_iter)
% membership generation
6: **for each** Ant
7: $\eta_j \leftarrow H = \sum_{i=1}^{n} H(i) - \sum_{j=1}^{n-1} I(j, j+1)$
8: $p_j^m(t) \leftarrow \frac{[\eta_j]\times[\tau_{ij}(t)]}{\sum_{i\in I_m}[\eta_i]\times[\tau_{ij}(t)]}$
9: $\tau_{ij}(t+1) = (1-\rho).\tau_{ij}(t) + \Delta\tau_{ij}(t)$
10:**end for each**
11: Best_sol_mem← **max(η_j)** % Best found value until iteration t
% non-membership generation
12: **for each** Ant
13 $\eta_j \leftarrow H = \sum_{i=1}^{n} H(i) - \sum_{j=1}^{n-1} I(j, j+1)$
14: $p_j^m(t) \leftarrow \frac{[\eta_j]\times[\tau_{ij}(t)]}{\sum_{i\in I_m}[\eta_i]\times[\tau_{ij}(t)]}$
15: $\tau_{ij}(t+1) = (1-\rho).\tau_{ij}(t) + \Delta\tau_{ij}(t)$
16:**end for each**
17: Best_sol_non-mem← **max(η_j)** % Best found value until iteration t
18: **End While**
18: Best _mem ← Best_sol_mem
19: Best _non-mem ← Best_sol_non-mem
20: indeterminacy← calculate-ind(Best _mem, Best _non-mem);
21: Draw(Best _mem, Best _non-mem, indeterminacy)
22: Draw_conversions_rate()
23: **Output** membership, non-membership and indeterminacy function, conversion rate.

Function calculate-ind($\mu_A(x)$, $\nu_A(x)$)
1: Input:($\mu_A(x)$, $\nu_A(x)$)
2: Output: indeterminacy
3: $0^- - [\mu_A(x) + \nu_A(x)] \leq \sigma_A(x) \leq 3^+ - [\mu_A(x) + \nu_A(x)]$
4: indeterminacy ←**Normalize ($\sigma_A(x)$);**
5: Return indeterminacy
5: End Fun

Fig. 5: Algorithm for the modelling neutrosophic variable using ACO

In membership function modelling, the total entropy function Eq. 13, 14 and 15 are calculated by the probability distribution P(x) of the variable data frequency in various partitions and the intersecting between them. The complement of probability distribution $1 - P(x)$ is utilized to measure the non-membership of variable data in different partitions. Therefore, the non-membership objective function will compute Eq. 13, 14 and 15 with the variable data frequency complement in different partitions and overlapping.

According to Eq.3 & 6, the summation of the membership, non-membership and indeterminacy values for the same instance is in the interval $[0^-, 3^+]$. Hence the indeterminacy function is declared by Eq. 16.

$$0^- - [\mu_A(x) + \nu_A(x)] \leq \sigma_A(x) \leq 3^+ - [\mu_A(x) + \nu_A(x)] \quad (16)$$

Where Eq. 9 states that the summation of the membership, non-membership and indeterminacy values for the same instance is in the interval $[0^-, 2^+]$. Hence, the indeterminacy function is defined as Eq. 17.

$$0^- - [\mu_A(x) + \nu_A(x)] \leq \sigma_A(x) \leq 2^+ - [\mu_A(x) + \nu_A(x)] \quad (17)$$

By finding the membership and non-membership definition of x, the indeterminacy function $\sigma_A(x)$ could be driven easily from Eq. 15 or 16. The value of the indeterminacy function should be in the interval $[0^- \ 1^+]$, hence the $\sigma_A(x)$ function is normalized according to Eq. 18.

$$Normalized_\sigma_A(x_i) = \frac{\sigma_A(x_i) - min(\sigma_A(x))}{max(\sigma_A(x)) - min(\sigma_A(x))} \quad (18)$$

where $\sigma_A(x_i)$ is the indeterminacy function for the value x_i. The flow chart and algorithm of the integrated framework is illustrated in figure 5 and 6 respectively.

4 Experimental Results

The present reality issues are brimming with vulnerability and indeterminism. The neutrosophic field is worried by picking up information with degrees of enrollment, indeterminacy and non-participation. Neutrosophic framework depends on various neutrosophic factors or variables. Unfortunately, the vast majority of the informational indexes accessible are normal numeric qualities or unmitigated characteristics. Henceforth, creating approaches for characterizing a neutrosophic set from the current informational indexes is required.

The membership capacity function of a neutrosophy variable, similar to the fuzzy variable, can take a few sorts. Triangle membership is very popular due to its simplicity and accuracy. Triangle function is characterized by various overlapping partitions. These subsets are characterized by support, limit and core

parameters. The most applicable parameter to a specific subset is the support which is the space of characterizing the membership degree. Finding the start and closure of a support over the universe of a variable could be an intriguing search issue suitable for optimization. Meta-heuristic search methodologies [22] give an intelligent procedure for finding ideal arrangement of solutions is any universe. ACO is a well-defined search procedure that mimics ants in discovering their sustenance. Figure 3 presents the ant as an individual in a population for upgrading a triangle membership function through the ACO procedure. The ACO utilizes the initial ant population and emphasizes to achieve ideal arrangement.

Table 1. Parameters of ACO

Maximum Number of Iterations	50
Population Size (number of ants)	10
Decaying rate	0.1

The total entropy given by Eq. 15 characterizes the heuristic desirability which affects the probabilistic transition rule of ants in the ACO algorithm. The probability distribution $P(x)$ presented in Eq. 13, 14 and 15 is used to calculate the total entropy function. The ACO parameters like Maximum Number of Iterations, Population Size, and pheromone decaying rate are presented in table 1.

The non-membership function means the falsity degree in the variables values. Hence, the complement of a data probability distribution $1 - P(x)$ is utilized to create the heuristic desirability of the ants in designing the non-membership function Eq. 13, 14 and 15.

The indeterminacy capacity of variable data is created by both membership and non-membership capacities of the same data using neutrosophic set declaration in section 2 and Eq. 16 or 17. Afterwards, Eq. 18 is used to normalize the indeterminacy capacity of the data. Through simulation, the ACO is applied by MATLAB , PC with Intel(R) Core (TM) CPU and 4 GB RAM. The simulation are implemented on the temperature variable from the Forest Fires data set created by: Paulo Cortez and Anbal Morais (Univ. Minho) [25]. The histogram of a random collection of the temperature data is shown in figure 7.

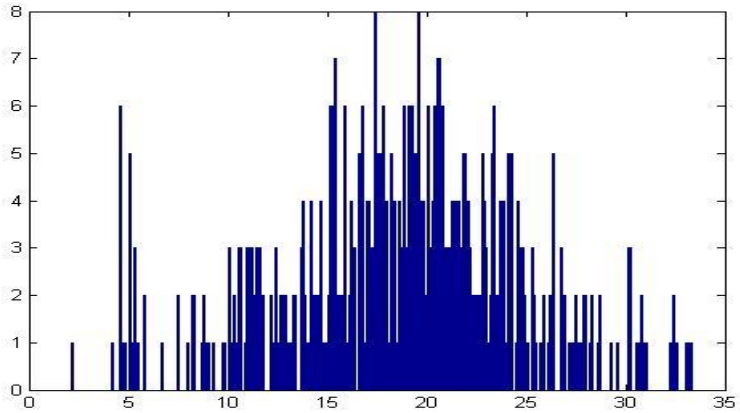


Fig. 6: Temperature Variable Histogram

Figures 7: a, b and c presents the resulting membership, non-membership and indeterminacy capacities produced by applying the ACO on a random collection of the temperature data.

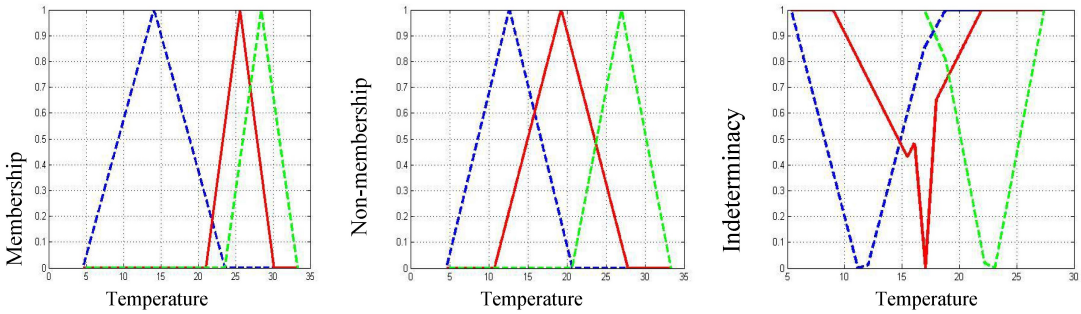


Fig. 7: a. Membership Function
b. Non-membership Function
c. Inderminacy Function

5 Conclusion

A proposed framework utilizing the ant colony optimization and the total entropy measure for mechanizing the design of neutrosophic variable is exhibited. The membership, non-membership and indeterminacy capacities are utilized to represent the neutrosophy idea. The enrollment or truth of subset could be conjured from total entropy measure. The fundamental system aggregates the

total entropy to the participation or truth subsets of a neutrosophic concept. The ant colony optimization is a meta-heuristic procedure which seeks the universe related to variable X to discover ideal segments or partitions parameters. The heuristic desirability of ants, for membership generation, is the total entropy based on the probability density function of random variable X. Thusly, the probability density complement is utilized to design non-membership capacity. The indeterminacy capacity is identified, as indicated by neutrosophic definition, by the membership and non-membership capacities. The results in light of ACO proposed system are satisfying. Therefore, the technique can be utilized as a part of data preprocessing stage within knowledge discovery system. Having sufficient data gathering, general neutrosophic variable outline for general data can be formulated.

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XIII

New Neutrosophic Sets via Neutrosophic Topological Spaces

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Abstract

In Geographical information systems (GIS) there is a need to model spatial regions with indeterminate boundary and under indeterminacy. The purpose of this chapter is to construct the basic concepts of the so-called "neutrosophic sets via neutrosophic topological spaces (NTs)". After giving the fundamental definitions and the necessary examples we introduce the definitions of neutrosophic open sets, neutrosophic continuity, and obtain several preservation properties and some characterizations concerning neutrosophic mapping and neutrosophic connectedness. Possible applications to GIS topological rules are touched upon.

Keywords

Logic, Set Theory, Topology, Neutrosophic set theory, Neutrosophic topology, Neutrosophic open set, Neutrosophic semi-open set, Neutrosophic continuous function.

1 Introduction

Neutrosophy has laid the foundation for a whole family of new mathematical theories generalizing both their classical and fuzzy counterparts, such as a neutrosophic set theory. In various recent papers, F. Smarandache generalizes intuitionistic fuzzy sets (IFSs) and other kinds of sets to neutrosophic sets (NSs). F. Smarandache also defined the notion of neutrosophic topology on the non-standard interval. Indeed, an intuitionistic fuzzy topology is not necessarilly a neutrosophic topology. Also, (Wang, Smarandache, Zhang, and

Sunderraman, 2005) introduced the notion of interval neutrosophic set, which is an instance of neutrosophic set and studied various properties. We study in this chapter relations between interval neutrosophic sets and topology. In this chapter, we introduce definitions of neutrosophic open sets. After given the fundamental definitions of neutrosophic set operations, we obtain several properties, and discussed the relationship between neutrosophic open sets and others, we introduce and study the concept of neutrosophic continuous functions. Finally, we extend the concepts of neutrosophic topological space.

2 Terminologies

We recollect some relevant basic preliminaries, and in particular, the work of Smarandache in [1, 2, 3], and Salama et al. [4, 5, 6, 7, 8, 9, 10, 11, 12, 13]. Smarandache introduced the neutrosophic components *T, I, F*, which represent the membership, indeterminacy, and non-membership values respectively, where $]^{-}0,1^{+}[$ is a non-standard unit interval. Hanafy and Salama et al. [10, 11] considered some possible definitions for basic concepts of the neutrosophic crisp set and its operations. We now improve some results by the following.

Definition 2.1 [24] Let *T, I, F* be real standard or nonstandard subsets of $]0^{-},1^{+}[$, with

Sup-T=t-sup, inf-T=t-inf

Sup-I=i-sup, inf-I=i-inf

Sup-F=f-sup, inf-F=f-inf

n-sup=t-sup+i-sup+f-sup

n-inf=t-inf+i-inf+f-inf.

T, I, F are called neutrosophic components.

We shall now consider some possible definitions for basic concepts of the neutrosophic set and its operations due to Salama et al.

Definition 2.2 [23] Let X be a non-empty fixed set. A neutrosophic set (NS for short) A is an object having the form

$$A = \{\langle x, \mu_A(x), \sigma_A(x), \gamma_A(x)\rangle : x \in X\}$$

where $\mu_A(x), \sigma_A(x)$, and $\gamma_A(x)$ which represent the degree of membership function (namely $\mu_A(x)$), the degree of indeterminacy (namely $\sigma_A(x)$), and the

degree of non-membership (namely $\gamma_A(x)$) respectively of each element $x \in X$ to the set A .

A neutrosophic $A = \{\langle x, \mu_A(x), \sigma_A(x), \gamma_A(x)\rangle : x \in X\}$ can be identified to an ordered triple $\langle \mu_A(x), \sigma_A(x), \gamma_A(x)\rangle$ in $]0^-, 1^+[$ on *X*.

Remark 2.3 [23] For the sake of simplicity, we shall use the symbol

$$A = \{x, \mu_A(x), \sigma_A(x), \gamma_A(x)\}$$

for the NS $A = \{\langle x, \mu_A(x), \sigma_A(x), \gamma_A(x)\rangle : x \in X\}$.

Definition 2.4 [4] Let $A = \langle \mu_A(x), \sigma_A(x), \gamma_A(x)\rangle$ *a NS on* X , then the complement of the set $A(C(A)$ for short, maybe defined as three kinds of complements

1. $C(A) = \{\langle x, 1-\mu_A(x), 1-\gamma_A(x)\rangle : x \in X\}$,
2. $C(A) = \{\langle x, \gamma_A(x), \sigma_A(x), \mu_A(x)\rangle : x \in X\}$,
3. $C(A) = \{\langle x, \gamma_A(x), 1-\sigma_A(x), \mu_A(x)\rangle : x \in X\}$,

One can define several relations and operations between GNSS as follows:

Since our main purpose is to construct the tools for developing neutrosophic set and neutrosophic topology, we must introduce the *NSS* 0_N and 1_N [23] in X as follows:

1- 0_N may be defined as four types:

1. $0_N = \{\langle x, 0, 0, 1\rangle : x \in X\}$ or
2. $0_N = \{\langle x, 0, 1, 1\rangle : x \in X\}$ or
3. $0_N = \{\langle x, 0, 1, 0\rangle : x \in X\}$ or
4. $0_N = \{\langle x, 0, 0, 0\rangle : x \in X\}$

2- 1_N may be defined as four types:

1. $1_N = \{\langle x, 1, 0, 0\rangle : x \in X\}$ or

2. $1_N = \{\langle x,1,0,1\rangle : x \in X\}$ or

3. $1_N = \{\langle x,1,1,0\rangle : x \in X\}$ or

4. $1_N = \{\langle x,1,1,1\rangle : x \in X\}$

Definition 2.5 [23] Let X be a non-empty set, and GNSS A and B in the form $A = \{x, \mu_A(x), \sigma_A(x), \gamma_A(x)\}$, $B = \{x, \mu_B(x), \sigma_B(x), \gamma_B(x)\}$, then we may consider two possible definitions for subsets $(A \subseteq B)$

$(A \subseteq B)$ may be defined as

1. Type 1:

$A \subseteq B \Leftrightarrow \mu_A(x) \le \mu_B(x), \sigma_A(x) \ge \sigma_B(x), \text{and} \gamma_A(x) \le \gamma_B(x)$ or

2. Type 1:

$A \subseteq B \Leftrightarrow \mu_A(x) \le \mu_B(x), \sigma_A(x) \ge \sigma_B(x), \text{and} \gamma_A(x) \ge \gamma_B(x)$.

Definition 2.6 [23] Let $\{A_j : j \in J\}$ be an arbitrary family of *NSS* in X, then

1. $\cap A_j$ may be defined as two types:

-Type 1: $\cap A_j = \langle x, \underset{j\in J}{\wedge} \mu_{Aj}(x), \underset{j\in J}{\wedge} \sigma_{Aj}(x), \underset{j\in J}{\vee} \gamma_{Aj}(x)\rangle$.

-Type 2: $\cap A_j = \langle x, \underset{j\in J}{\wedge} \mu_{Aj}(x), \underset{j\in J}{\vee} \sigma_{Aj}(x), \underset{j\in J}{\vee} \gamma_{Aj}(x)\rangle$.

2. $\cup A_j$ may be defined as two types:

-Type 1: $\cup A_j = \langle x, \underset{j\in J}{\vee} \mu_{Aj}(x), \underset{j\in J}{\vee} \sigma_{Aj}(x), \underset{j\in J}{\wedge} \gamma_{Aj}(x)\rangle$.

-Type 2: $\cup A_j = \langle x, \underset{j\in J}{\vee} \mu_{Aj}(x), \underset{j\in J}{\wedge} \sigma_{Aj}(x), \underset{j\in J}{\wedge} \gamma_{Aj}(x)\rangle$.

Definition 2.7 [25] A neutrosophic topology (NT for short) and a non empty set X is a family τ of neutrosophic subsets in X satisfying the following axioms

1. $0_N, 1_N \in \tau$

2. $G_1 \cap G_2 \in \tau$ for any $G_1, G_2 \in \tau$

3. $\cup G_i \in \tau$, $\forall\{G_i \mid j \in J\} \subseteq \tau$.

In this case the pair (X,τ) is called a neutrosophic topological space (NTS for short) and any neutrosophic set in τ is known as neutrosophic open set (NOS for short) in X. The elements of τ are called open neutrosophic sets, A neutrosophic set F is closed if and only if it $C(F)$ is neutrosophic open [26-30].

Note that for any NTS A in (X,τ), we have $Cl(A^c) = [Int(A)]^c$ and $Int(A^c) = [Cl(A)]^c$.

Example 2.8 [4] Let $X = \{a,b,c,d\}$, and $A = \{x, \mu_A(x), \sigma_A(x), \gamma_A(x)\}$

$A = \{\langle x,0.5,0.5,0.4\rangle : x \in X\}$

$B = \{\langle x,0.4,0.6,0.8\rangle : x \in X\}$

$D = \{\langle x,0.5,0.6,0.4\rangle : x \in X\}$

$C = \{\langle x,0.4,0.5,0.8\rangle : x \in X\}$

Then the family $\tau = \{0_n, 1_n, A, B, C, D\}$ of NSs in X is neutrosophic topology on X.

Definition 2.9 [23] Let (x,τ) be NTs and $A = \{x, \mu_A(x), \sigma_A(x), \gamma_A(x)\}$ be a NS in X.

Then the neutrosophic closure and neutrosophic interior of A are defined by

1. $NCL(A) = \cap\{K : K \text{ is a NCS in X and } A \subseteq K\}$
2. $NInt(A) = \cup\{G : G \text{ is a NOS in X and } G \subseteq A\}$

It can be also shown that $NCl(A)$ is NCS and $NInt(A)$ is a NOS in X

1. A is in X if and only if $NCl(A)$.
2. A is NCS in X if and only if $NInt(A) = A$.

Proposition 2.10 [23] Let (x,τ) be a NTS and A, B be two neutrosophic sets in X. Then the following properties hold:

1. $NInt(A) \subseteq A$,
2. $A \subseteq NCl(A)$,
3. $A \subseteq B \Rightarrow NInt(A) \subseteq NInt(B)$,
4. $A \subseteq B \Rightarrow NCl(A) \subseteq NCl(B)$,
5. $NCL(NCL(A)) = NCL(A)$

   $NInt(NInt(A)) = NInt(A)$,
6. $NInt(A \cup B) = NInt(A) \cup NInt(B)$

   $NCl(A \cap B) = NCl(A) \cap NCl(B)$,
7. $NCl(A) \cup NCl(B) = NInt(A \cup B)$,

Definition 2.11 [23] Let $A = \{\mu_A(x), \sigma_A(x), \gamma_A(x)\}$ be a neutrosophic open sets and $B = \{\mu_B(x), \sigma_B(x), \gamma_B(x)\}$ be a neutrosophic set on a neutrosophic topological space (X, τ) then

1. A is called neutrosophic regular open iff $A = NInt(NCl(A))$.
2. If $B \in NCS(X)$ then B is called neutrosophic regular closed iff $A = NCl(NInt(A))$.

3 Neutrosophic Openness

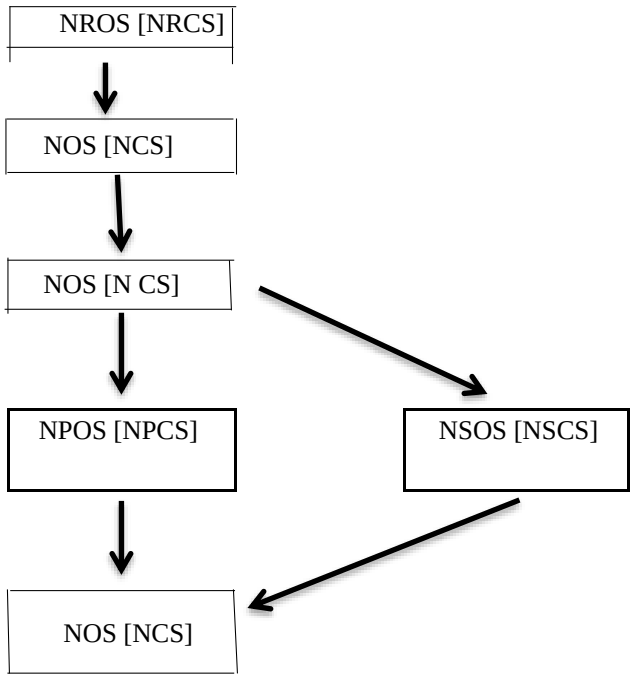
Definition 3.1 A neutrosophic set (Ns) A in a neutrosophic topology (X, τ) is called

1. Neutrosophic semiopen set $(NSOS)$ if $A \subseteq NCl(NInt(A))$,
2. Neutrosophic preopen set $(NPOS)$ if $A \subseteq NInt(NCl(A))$,
3. Neutrosophic α -open set $(N\alpha OS)$ if $A \subseteq NInt(NCl(NInt(A)))$
4. Neutrosophic β -open set $(N\beta OS)$ if $A \subseteq NCl(NInt(NCl(A)))$

An (Ns) A is called neutrosophic semi-closed set, neutrosophic α -closed set, neutrosophic pre-closed set, and neutrosophic regular closed set, respectively (NSCS, Nα CS, NPCS, and NRCS, resp.), if the complement of A is a NSOS, Nα OS, NPOS, and NROS, respectively.

Definition 3.2 In the following diagram, we provide relations between various types of neutrosophic openness (neutrosophic closedness): 0pt

Remark 3.3 From above the following implication and none of these implications is reversible as shown by examples given below



Reverse implications are not true in the above diagram. The following is a characterization of a Nα OS.

Example 3.4 Let $X = \{a, b, c\}$ and:

$A = \langle (0.5,0.5,0.5),(0.4,0.5,0.5),(0.4,0.5,0.5)\rangle$,

$B = \langle (0.3,0.4,0.4),(0.7,0.5,0.5),(0.3,0.4,0.4)\rangle$.

Then $\tau = \{0_N, 1_N, A, B\}$ is a neutrosophic topology on X . Define the two neutrosophic closed sets C_1 and C_2 as follows,

$C_1 = \langle (0.5,0.5,0.5),(0.6,0.5,0.5),(0.6,0.5,0.5)\rangle$,

$C_2 = \langle (0.7,0.6,0.6),(0.3,0.5,0.5),(0.7,0.6,0.6)\rangle$.

Then the set A is neutrosophic open set (NOs) but not neutrosophic regular open set (NROs) since $A \neq NInt(NCl(A))$, and since $A \subseteq NInt(NCl(NInt(A)))$ where the $NInt(NCl(NInt(A)))$ is equal to:

$$\langle (0.5,0.5,0.5),(0.3,0.5,0.5),(0.7,0.6,0.6)\rangle$$

so that A is neutrosophic α -open set (Nα Os).

Example 3.5 Let $X = \{a,b,c\}$ and:

$A = \langle (0.5,0.5,0.5),(0.4,0.5,0.5),(0.4,0.5,0.5)\rangle$,

$B = \langle (0.3,0.4,0.4),(0.7,0.5,0.5),(0.3,0.4,0.4)\rangle$, and

$C = \langle (0.5,0.5,0.5),(0.4,0.5,0.5),(0.5,0.5,0.5)\rangle$.

Then $\tau = \{0_N, 1_N, A, B\}$ is a neutrosophic topology on X . Define the two neutrosophic closed sets C_1 and C_2 as follows:

$C_1 = \langle (0.5,0.5,0.5),(0.6,0.5,0.5),(0.6,0.5,0.5)\rangle$,
$C_2 = \langle (0.7,0.6,0.6),(0.3,0.5,0.5),(0.7,0.6,0.6)\rangle$.

Then the set C is neutrosophic semi open set (NSOs), since

$C \subseteq NCl(NInt(C))$,

where $NCl(NInt(C)) = \langle (0.5,0.5,0.5),(0.3,0.5,0.5),(0.7,0.6,0.6)\rangle$ but not neutrosophic α -open set (Nα Os) since $C \not\subseteq NInt(NCl(NInt(C)))$ where the $NInt(NCl(NInt(C)))$ is equal $\langle (0.5,0.5,0.5),(0.4,0.5,0.5),(0.3,0.4,0.4)\rangle$, in the sense of $A \subseteq B \Leftrightarrow \mu_A(x) \leq \mu_B(x), \sigma_A(x) \geq \sigma_B(x), \text{and } \gamma_A(x) \leq \gamma_B(x)$.

Example 3.6 Let $X = \{a,b,c\}$ and:

$A = \langle (0.4,0.5,0.4),(0.5,0.5,0.5),(0.4,0.5,0.4)\rangle$,

$B = \langle (0.7,0.6,0.5),(0.3,0.4,0.5),(0.3,0.4,0.4)\rangle$, and

$C = \langle (0.5,0.5,0.5),(0.5,0.5,0.5),(0.5,0.5,0.5)\rangle$.

Then $\tau = \{0_N, 1_N, A, B\}$ is a neutrosophic topology on X . Define the two neutrosophic closed sets C_1 and C_2 as follows:

$C_1 = \langle (0.6,0.5,0.6),(0.5,0.5,0.5),(0.6,0.5,0.5)\rangle$,

$C_2 = \langle (0.3,0.4,0.5),(0.7,0.6,0.5),(0.7,0.6,0.5)\rangle$.

Then the set C is neutrosophic preopen set *(NPOs)*, since $C \subseteq NInt(NCl(C))$, where $NInt(NCl(C)) = \langle(0.7,0.6,0.5),(0.5,0.5,0.5),(0.3,0.4,0.5)\rangle$ but not neutrosophic α -open set ($N\alpha Os$) since $C\, NInt(NCl(NInt(C)))$ where the $NInt(NCl(NInt(C)))$ is equal $\langle(0,0,0),(1,1,1),(0,0,0)\rangle$.

Example 3.7 Let $X = \{a,b,c\}$ and:

$A = \langle(0.5,0.5,0.5),(0.4,0.5,0.5),(0.4,0.5,0.5)\rangle$,

$B = \langle(0.3,0.4,0.4),(0.7,0.5,0.5),(0.3,0.4,0.4)\rangle$, and

$C = \langle(0.3,0.3,0.3),(0.4,0.5,0.5),(0.3,0.4,0.4)\rangle$.

Then $\tau = \{0_N, 1_N, A, B\}$ is a neutrosophic topology on X. Define the two neutrosophic closed sets C_1 and C_2 as follows,

$C_1 = \langle(0.5,0.5,0.5),(0.6,0.5,0.5),(0.6,0.5,0.5)\rangle$,

$C_2 = \langle(0.7,0.6,0.6),(0.3,0.5,0.5),(0.7,0.6,0.6)\rangle$.

Then the set C is neutrosophic β -open set *(Nβ Os)*, since $C \subseteq NCl(NInt(NCl(C)))$, where

$$NCl(NInt(NCl(A))) = \langle(0.7,0.6,0.6),(0.3,0.5,0.5),(0.7,0.6,0.6)\rangle,$$

but not neutrosophic pre-open set (*NPOs*) neither neutrosophic semi-open set (*NSOs*) since $C\, NCl(NInt(C))$ where the $NCl(NInt(C))$ is equal $\langle(0.5,0.5,0.5),(0.3,0.5,0.5),(0.7,0.6,0.6)\rangle$

Let (X,τ) be NTS and $A = \{A_1, A_2, A_3\}$ be a NS in X. Then the *-neutrosophic closure of A ($*-NCl(A)$ for short) and *-neutrosophic interior ($*-NInt(A)$ for short) of A are defined by

1. $\alpha NCl(A) = \cap\{K : is a NRCS in X and A \subseteq K\}$,
2. $\alpha NInt(A) = \cup\{G : G is a NROS in X and G \subseteq A\}$,
3. $pNCl(A) = \cap\{K : is a NPCS in X and A \subseteq K\}$,
4. $pNInt(A) = \cup\{G : G is a NPOS in X and G \subseteq A\}$,
5. $sNCl(A) = \cap\{K : is a NSCS in X and A \subseteq K\}$,
6. $sNInt(A) = \cup\{G : G is a NSOS in X and G \subseteq A\}$,

7. $\beta NCl(A) = \cap\{K : is a NC\beta CS in X and A \subseteq K\}$,
8. $\beta NInt(A) = \cup\{G : G is a N\beta OS in X and G \subseteq A\}$,
9. $rNCl(A) = \cap\{K : is a NRCS in X and A \subseteq K\}$,
10. $rNInt(A) = \cup\{G : G is a NROS in X and G \subseteq A\}$.

Theorem 3.8 *A Ns* A in a *NTs* (X, τ) is a $N\alpha$ *OS* if and only if it is both *NSOS* and *NPOS.*

Proof. Necessity follows from the diagram given above. Suppose that A is both a *NSOS* and a *NPOS.* Then $A \subseteq NCl(NInt(A))$, and so

$$NCl(A) \subseteq NCl(NCl(NInt(A))) = NCl(NInt(A))$$

It follows that $A \subseteq NInt(NCl(A)) \subseteq NInt(NCl(NInt(A)))$, so that A is a $N\alpha$ *OS*. We give condition(s) for a NS to be a $N\alpha$ *OS*.

Proposition 3.9 Let (X, τ) be a neutrosophic topology space *NTs*. Then arbitrary union of neutrosophic α -open sets is a neutrosophic α -open set, and arbitrary intersection of neutrosophic α -closed sets is a neutrosophic α -closed set.

Proof. Let $A = \{\langle x, \mu_{A_i}, \sigma_{A_i}, \gamma_{A_i} \rangle : i \in \Lambda\}$ be a collection of neutrosophic α -open sets. Then, for each $i \in \Lambda$, $A_i \subseteq NInt(NCl(NInt(A_i)))$. Its follows that

$$\bigcup A_i \subseteq \bigcup NInt(NCl(NInt(A_i))) \subseteq NInt(\bigcup NCl(NInt(A_i)))$$

$$= NInt(NCl(\bigcup NInt(A_i))) \subseteq NInt(NCl(NInt(\bigcup A_i)))$$

Hence $\bigcup A_i$ is a neutrosophic α -open set. The second part follows immediately from the first part by taking complements.

Having shown that arbitrary union of neutrosophic α -open sets is a neutrosophic α -open set, it is natural to consider whether or not the intersection of neutrosophic α -open sets is a neutrosophic α -open set, and the following example shown that the intersection of neutrosophic α -open sets is not a neutrosophic α -open set.

Example 3.10 Let $X = \{a, b, c\}$ and

$A = \langle (0.5,0.5,0.5),(0.4,0.5,0.5),(0.4,0.5,0.5)\rangle$,
$B = \langle (0.3,0.4,0.4),(0.7,0.5,0.5),(0.3,0.4,0.4)\rangle$.

Then $\tau = \{0_N, 1_N, A, B\}$ is a neutrosophic topology on X . Define the two neutrosophic closed sets C_1 and C_2 as follows,

$C_1 = \langle (0.5,0.5,0.5),(0.6,0.5,0.5),(0.6,0.5,0.5)\rangle$,
$C_2 = \langle (0.7,0.6,0.6),(0.3,0.5,0.5),(0.7,0.6,0.6)\rangle$.

Then the set A and B are neutrosophic α -open set (N α Os) but $A \cap B$ is not neutrosophic α -open set. In fact $A \cap B$ is given by $\langle (0.3,0.4,0.4),(0.4,0.5,0.5),(0.4,0.5,0.5)\rangle$,

and $NInt(NCl(NInt(A \cap B))) = \langle (0.5,0.5,0.5),(0.7,0.5,0.5),(0.3,0.4,0.4)\rangle$,

so $A \cap BNInt(NCl(NInt(A \cap B)))$.

Theorem 3.11 Let A be a (Ns) in a neutrosophic topology space NTs (X,τ). If B is a *NSOS* such that $B \subseteq A \subseteq NInt(NCl(B))$, then A is a $N\alpha$ *OS.*

Proof. Since B is a NSOS, we have $B \subseteq NCl(NInt(B))$. Thus, $A \subset NInt(NCl(A)) \subseteq NInt(NCl(NCl(NInt(B)))) = NInt(NCl(NInt(B))) \subseteq NInt(NCl(NInt(A)))$. and so is a a $N\alpha$ *OS*

Proposition 3.12 In neutrosophic topology space NTs (X,τ), a neutrosophic α -closed *($N\alpha$ Cs) if* and only if $A = \alpha NCl(A)$.

Proof. Assume that A is neutrosophic α -closed set. Obviously,

$A \in \{B_i \mid B_i \text{isaneutrosophic } -closed\, set\, and\, A \subseteq B_i\}$,

and also

$A = \{B_i \mid B_i \text{isaneutrosophic } -closed\, set\, and\, A \subseteq B_i\}$,

$= \alpha NCl(A)$.

Conversely suppose that $A = \alpha NCl(A)$, which shows that

$A \in \{B_i \mid B_i \text{isaneutrosophic } -closed\, set\, and\, A \subseteq B_i\}$.

Hence A is neutrosophic α -closed set.

Theorem 3.13 A neutrosophic set A in a *NTs* X is neutrosophic α -open (resp., neutrosophic preopen) if and only if for every $N\alpha Os_{p(\alpha,\beta)} \in A$, there exists a N$\alpha$ Os (resp., NPOs) $B_{p(\alpha,\beta)}$ such that $p(\alpha,\beta) \in B_{p(\alpha,\beta)} \subseteq A$.

Proof. If A is a Nα Os (resp., NPOs), then we may take $B_{p(\alpha,\beta)} = A$ for every $p(\alpha,\beta) \in A$.

Conversely assume that for every NP $p(\alpha,\beta) \in A$, there exists a Nα Os (resp., NPOs) $B_{p(\alpha,\beta)}$ such that $p(\alpha,\beta) \in B_{p(\alpha,\beta)} \subseteq A$. Then,

$$A = \bigcup\{p(\alpha,\beta) \mid p(\alpha,\beta) \in A\} \subseteq \bigcup\{B_{p(\alpha,\beta)} \mid p(\alpha,\beta) \in A\} \subseteq A,$$

and so $$A = \bigcup\{B_{p(\alpha,\beta)} \mid p(\alpha,\beta) \in A\},$$

which is a Nα Os (resp., NPOs) by Proposition 3.9.

Proposition 3.14 In a *NTs* (X,τ), the following hold for neutrosophic α -closure:

1. $\alpha NCl(0_{\sim}) = 0_{\sim}$.
2. $\alpha NCl(A)$ is neutrosophic α -closed in (X,τ) for every Ns in A.
3. $\alpha NCl(A) \subseteq \alpha NCl(B)$ whenever $A \subseteq B$ for every Ns A and B in X.
4. $\alpha NCl(\alpha NCl(A)) = \alpha NCl(A)$ for every Ns A in X.

Proof. The proof is easy.

4 Neutrosophic Continuous Mapping

Definition 4.1 [25] Let (X,τ_1) and (Y,τ_2) be two NTSs, and let $f: X \to Y$ be a function. Then f is said to be strongly N -continuous iff the inverse image of every NOS in τ_2 is a NOS in τ_1.

Definition 4.2 [25] Let (X,τ_1) and (Y,τ_2) be two NTSs, and let $f: X \to Y$ be a function. Then f is said to be continuous iff the preimage of each NS in τ_2 is a NS in τ_1.

Example 4.3 [25] Let $X = \{a, b, c\}$ and $Y = \{a, b, c\}$. Define neutrosophic sets A and B as follows:

$A = \langle (0.4, 0.4, 0.5), (0.2, 0.4, 0.3), (0.4, 0.4, 0.5) \rangle$,

$B = \langle (0.4, 0.5, 0.6), (0.3, 0.2, 0.3), (0.4, 0.5, 0.6) \rangle$.

Then the family $\tau_1 = \{0_N, 1_N, A\}$ is a neutrosophic topology on X and $\tau_2 = \{0_N, 1_N, B\}$ is a neutrosophic topology on Y. Thus (X, τ_1) and (Y, τ_2) are neutrosophic topological spaces. Define $f : (X, \tau_1) \to (Y, \tau_2)$

as $f(a) = b$, $f(b) = a$, $f(c) = c$.

Clearly f is N-continuous.

Now f is not neutrosophic continuous, since $f^{-1}(B) \notin \tau$ for $B \in \tau_2$.

Definition 4.4 Let f be a mapping from a NTS (X, τ) to a NTS (Y, κ). Then f is called

1. a neutrosophic α-continuous mapping if $f^{-1}(B)$ is a Nα Os in X for every NOs B in Y.
2. a neutrosophic pre-continuous mapping if $f^{-1}(B)$ is a NPOs in X for every NOs B in Y.
3. a neutrosophic semi-continuous mapping if $f^{-1}(B)$ is a NSOs in X for every NOs B in Y.
4. a neutrosophic β-continuous mapping if $f^{-1}(B)$ is a Nβ Os in X for every NOs B in Y.

Theorem 4.5 For a mapping f from a *NTS* (X, τ) to a *NTS* (Y, κ), the following are equivalent.

1. f is neutrosophic pre-continuous.
2. $f^{-1}(B)$ is *NPCs* in X for every NCs B in Y.
3. $NCl(NInt(f^{-1}(A))) \subseteq f^{-1}(NCl(A))$ for every neutrosophic set A in Y.

Proof. $(1) \Rightarrow (2)$ The proof is straightforward.

$(2) \Rightarrow (3)$ Let A be a NS in Y. Then $NCl(A)$ is neutrosophic closed. It follows from (2) that $f^{-1}(NCl(A))$ is a *NPCS* in X so that

$$NCl(NInt(f^{-1}(A))) \subseteq NCl(NInt(f^{-1}(NCl(A)))) \subseteq f^{-1}(NCl(A)).$$

$(3) \Rightarrow (1)$ Let A be a *NOS* in Y. Then $\overline{A}$ is a *NCS* in Y, and so

$$NCl(NInt(f^{-1}(\overline{A}))) \subseteq f^{-1}(NCl(\overline{A})) = f^{-1}(\overline{A}).$$

This implies that

$$\overline{NInt(NCl(f^{-1}(A)))} = NCl(\overline{NCl(f^{-1}(A))}) = NCl(NInt(\overline{f^{-1}(A)}))$$
$$= NCl(NInt(f^{-1}(\overline{A}))) \subseteq f^{-1}(\overline{A}) = \overline{f^{-1}(A)},$$

and thus $f^{-1}(A) \subseteq NInt(NCl(f^{-1}(A)))$. Hence $f^{-1}(A)$ is a *NPOS* in X, and f is neutrosophic pre-continuous.

Theorem 4.6 Let f be a mapping from a *NTS* (X, τ) to a *NTS* (Y, κ) that satisfies

$NCl(NInt(NCl(f^{-1}(B)))) \subseteq f^{-1}(NCl(B))$, for every NS B in Y. Then f is neutrosophic α-continuous.

Proof. Let B be a NOS in Y. Then B is a *NCS* in Y, which implies from hypothesis that

$$NCl(NInt(NCl(f^{-1}(\overline{B})))) \subseteq f^{-1}(NCl(\overline{B})) = f^{-1}(\overline{B}).$$

It follows that

$$\overline{NInt(NCl(NInt(f^{-1}(B))))} = NCl(\overline{NCl(NInt(f^{-1}(B)))})$$
$$= NCl(NInt(\overline{NInt(f^{-1}(B))}))$$
$$= NCl(NInt(NCl(\overline{f^{-1}(B)})))$$
$$= NCl(NInt(NCl(f^{-1}(\overline{B})))) \subseteq f^{-1}(\overline{B})$$

$$= \overline{f^{-1}(B)}$$

so that $f^{-1}(B) \subseteq NInt(NCl(NInt(f^{-1}(B))))$. This shows that $f^{-1}(B)$ is a N α OS in X. Hence, f is neutrosophic α -continuous.

Definition 4.7 Let $p(\alpha,\beta)$ be a *NP* of a *NTS* (X,τ). A NS A of X is called a neutrosophic neighborhood (*NH*) of $p(\alpha,\beta)$ if there exists a NOS B in X such that $p(\alpha,\beta) \in B \subseteq A$.

Theorem 4.8 Let f be a mapping from a NTS (X,τ) to a *NTS* (Y,κ). Then the following assertions are equivalent.

1. f is neutrosophic pre-continuous.
2. For each *NP* $p(\alpha,\beta) \in X$ and every *NH* A of $f(p(\alpha,\beta))$, there exists a *NPOS* B in X such that $p(\alpha,\beta) \in B \subseteq f^{-1}(A)$.
3. For each *NP* $p(\alpha,\beta) \in X$ and every NH A of $f(p(\alpha,\beta))$, there exists a *NPOS* B in X such that $p(\alpha,\beta) \in B$ and $f(B) \subseteq A$.

Proof. $(1) \Rightarrow (2)$ Let $p(\alpha,\beta)$ be a NP in X and let A be a NH of $f(p(\alpha,\beta))$. Then there exists a NOS B in Y such that $f(p(\alpha,\beta)) \in B \subset A$. Since f is neutrosophic pre-continuous, we know that $f^{-1}(B)$ is a NPOS in X and

$$p(\alpha,\beta) \in f^{-1}(f(p(\alpha,\beta))) \subseteq f^{-1}(B) \subseteq f^{-1}(A).$$

Thus (2) is valid.

$(2) \Rightarrow (3)$ Let $p(\alpha,\beta)$ be a NP in X and let A be a NH of $f(p(\alpha,\beta))$. The condition (2) implies that there exists a NPOS B in X such that $p(\alpha,\beta) \in B \subseteq f^{-1}(A)$ so that $p(\alpha,\beta) \in B$ and $f(B) \subseteq f(f^{-1}(A)) \subseteq A$. Hence (3) is true.

$(3) \Rightarrow (1)$. Let B be a NOS in Y and let $p(\alpha,\beta) \in f^{-1}(B)$. Then $f(p(\alpha,\beta)) \in B$, and so B is a NH of $f(p(\alpha,\beta))$ since B is a NOS. It follows from (3) that there exists a NPOS A in X such that $p(\alpha,\beta) \in A$ and $f(A) \subseteq B$ so that,

$$p(\alpha,\beta)\in A\subseteq f^{-1}(f(A))\subseteq f^{-1}(B).$$

Applying Theorem 3.13 induces that $f^{-1}(B)$ is a NPOS in X. Therefore, f is neutrosophic pre-continuous.

Theorem 4.9 Let f be a mapping from a *NTS* (X,τ) to a *NTS* (Y,κ). Then the following assertions are equivalent.

1. f is neutrosophic α-continuous.
2. For each *NP* $p(\alpha,\beta)\in X$ and every *NH* A of $f(p(\alpha,\beta))$, there exists a *N*α *OS* B in X such that $p(\alpha,\beta)\in B\subseteq f^{-1}(A)$.
3. For each *NP* $p(\alpha,\beta)\in X$ and every NH A of $f(p(\alpha,\beta))$, there exists a *N*α *OS* B in X such that $p(\alpha,\beta)\in B$ and $f(B)\subseteq A$.

Proof. $(1)\Rightarrow(2)$ Let $p(\alpha,\beta)$ be a NP in X and let A be a NH of $f(p(\alpha,\beta))$. Then there exists a NOS C in Y such that $f(p(\alpha,\beta))\in B\subset A$. Since f is neutrosophic α-continuous, $B=f^{-1}(C)$ is a NPOS in X and

$$p(\alpha,\beta)\in f^{-1}(f(p(\alpha,\beta)))\subseteq B=f^{-1}(C)\subseteq f^{-1}(A).$$

Thus (2) is valid.

$(2)\Rightarrow(3)$ Let $p(\alpha,\beta)$ be a NP in X and let A be a NH of $f(p(\alpha,\beta))$. Then there exists a Nα OS B in X such that $p(\alpha,\beta)\in B\subseteq f^{-1}(A)$ by (2). Thus, we have $p(\alpha,\beta)\in B$ and $f(B)\subseteq f(f^{-1}(A))\subseteq A$. Hence (3) is valid.

$(3)\Rightarrow(1)$. Let B be a *NOS* in Y and we take $p(\alpha,\beta)\in f^{-1}(B)$. Then $f(p(\alpha,\beta))\in f(f^{-1}(B))\subseteq B$, Since B is NOS, it follows that B is a NH of $f(p(\alpha,\beta))$ so from (3), there exists a Nα OS A such that $p(\alpha,\beta)\in A$ and $f(A)\subset B$ so that,

$$p(\alpha,\beta)\in A\subseteq f^{-1}(f(A))\subseteq f^{-1}(B).$$

Using Theorem 3.13 induces that $f^{-1}(B)$ is a $N\alpha$ *OS* in X. Therefore, f is neutrosophic α -continuous.

Combining Theorems 4.6 and 4.9, we have the following characterization of neutrosophic α -continuous.

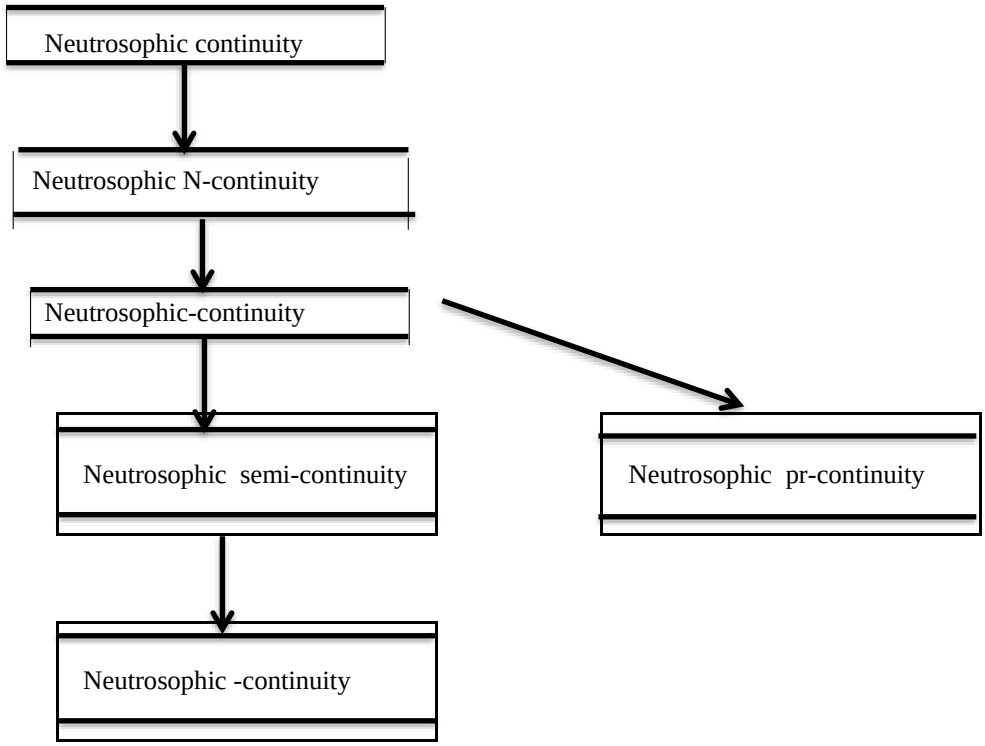
Theorem 4.10 Let f be a mapping from a *NTS* (X,τ) to a *NTS* (Y,κ). Then the following assertions are equivalent.

1. f is neutrosophic α -continuous.
2. If C is a *NCS* in Y, then $f^{-1}(C)$ is a Nα CS in X.
3. $NCl(NInt(NCl(f^{-1}(B))))\subseteq f^{-1}(NCl(B))$ for every *NS* B in Y.
4. For each *NP* $p(\alpha,\beta)\in X$ and every *NH* A of $f(p(\alpha,\beta))$, there exists a $N\alpha$ *OS* B such that $p(\alpha,\beta)\in B\subseteq f^{-1}(A)$.
5. For each *NP* $p(\alpha,\beta)\in X$ and every *NH* A of $f(p(\alpha,\beta))$, there exists a $N\alpha$ *OS* B such that $p(\alpha,\beta)\in B$ and $f(B)\subseteq A$.

Some aspects of neutrosophic continuity, neutrosophic N-continuity, neutrosophic strongly neutrosophic continuity, neutrosophic perfectly neutrosophic continuity, neutrosophic strongly *N*-continuity are studied in [25] as well as in several papers. The relation among these types of neutrosophic continuity is given as follows, where N means neutrosophic:

Example 4.11 Let $X=Y=\{a,b,c\}$. Define neutrosophic sets A and B as follows $A=\langle(0.5,0.5,0.5),(0.4,0.5,0.5),(0.4,0.5,0.5)\rangle$,
$B=\langle(0.3,0.4,0.4),(0.7,0.5,0.5),(0.3,0.4,0.4)\rangle$,
$C=\langle(0.5,0.5,0.5),(0.4,0.5,0.5),(0.5,0.5,0.5)\rangle$ and
$D=\langle(0.4,0.5,0.5),(0.5,0.5,0.5),(0.5,0.5,0.5)\rangle$. Then the family $\tau_1=\{0_N,1_N,A,B\}$ is a neutrosophic topology on X and $\tau_2=\{0_N,1_N,D\}$ is a neutrosophic topology on Y. Thus (X,τ_1) and (Y,τ_2) are neutrosophic topological spaces. Define $f:(X,\tau_1)\to(Y,\tau_2)$ as $f(a)=b$, $f(b)=a$, $f(c)=c$. Clearly f is neutrosophic semi-continuous, but not neutrosophic α -

continuous, since $f^{-1}(D) = C$ not not neutrosophic α -open set, i.e $C \not\subseteq NInt(NCl(NInt(C)))$ where the $NInt(NCl(NInt(C)))$ is equal $\langle (0.5,0.5,0.5),(0.4,0.5,0.5),(0.3,0.4,0.4)\rangle$.



The reverse implications are not true in the above diagram in general as the following example.

Example 4.12 Let $X = Y = \{a,b,c\}$ and

$A = \langle (0.4,0.5,0.4),(0.5,0.5,0.5),(0.4,0.5,0.4)\rangle$,

$B = \langle (0.7,0.6,0.5),(0.3,0.4,0.5),(0.3,0.4,0.4)\rangle$, and

$C = \langle (0.5,0.5,0.5),(0.5,0.5,0.5),(0.5,0.5,0.5)\rangle$.

Then $\tau_1 = \{0_N, 1_N, A, B\}$ is a neutrosophic topology on X and $\tau_2 = \{0_N, 1_N, C\}$ is a neutrosophic topology on Y. Thus (X,τ_1) and (Y,τ_2) are neutrosophic topological spaces. Define $f:(X,\tau_1)\to(Y,\tau_2)$ as identity

function. Then f is neutrosophic pre-continuous but not neutrosophic α-continuous, since $f^{-1}(C) = C$ is neutrosophic pre open set (*NPOs*) but not neutrosophic α-open set (*N*α *Os*).

Example 4.13 Let $X = Y = \{a, b, c\}$. Define neutrosophic sets A and B as follows $A = \langle (0.5, 0.5, 0.5), (0.4, 0.5, 0.5), (0.4, 0.5, 0.5) \rangle$, $B = \langle (0.3, 0.4, 0.4), (0.7, 0.5, 0.5), (0.3, 0.4, 0.4) \rangle$, and $D = \langle (0.3, 0.4, 0.4), (0.3, 0.3, 0.3), (0.4, 0.5, 0.5) \rangle$. $\tau_1 = \{0_N, 1_N, A, B\}$ is a neutrosophic topology on X and $\tau_2 = \{0_N, 1_N, D\}$ is a neutrosophic topology on Y. Define $f : (X, \tau_1) \to (Y, \tau_2)$ as $f(a) = c$, $f(b) = a$, $f(c) = b$. Clearly f is neutrosophic β-continuous, but not neutrosophic pre-continuous neither neutrosophic semi-continuous since $f^{-1}(D) = \langle (0.3, 0.3, 0.3), (0.4, 0.5, 0.5), (0.3, 0.4, 0.4) \rangle = C$ is neutrosophic β-open set (N β Os), since $C \subseteq NCl(NInt(NCl(C)))$, where $NCl(NInt(NCl(A))) = \langle (0.7, 0.6, 0.6), (0.3, 0.5, 0.5), (0.7, 0.6, 0.6) \rangle$, but not neutrosophic pre-open set (*NPOs*) neither neutrosophic semi-open set (*NSOs*) since $CNCl(NInt(C))$ where the $NCl(NInt(C))$ is equal $\langle (0.5, 0.5, 0.5), (0.3, 0.5, 0.5), (0.7, 0.6, 0.6) \rangle$.

Theorem 4.14 *Let* f be a mapping from NTS (X, τ_1) to *NTS* (X, τ_2). If f is both neutrosophic pre-continuous and neutrosophic semi-continuous, neutrosophic α-continuous.

Proof. Let B be an NOS in Y. Since f is both neutrosophic pre-continuous and neutrosophic semi-continuous, $f^{-1}(B)$ is both *NPOS* and *NSOS* in X. It follows from Theorem 3.8 that $f^{-1}(B)$ is a *N*α *OS* in X so that f is neutrosophic α-continuous.

5 Conclusion

In this chapter, we have introduced neutrosophic α-open sets, neutrosophic semi-open sets, and studied some of its basic properties. Also we study the relationship between the newly introduced sets namely introduced neutrosophic α-open sets and some of neutrosophic open sets that already exists. In this chapter also, we presented the basic definitions of the neutrosophic α-topological space and the neutrosophic α-compact space with some of their characterizations were deduced. Furthermore, we constructed a neutrosophic α-continuous function, with a study of a number its properties. Many different adaptations, tests, and experiments have been left for the future due to lack of time. There are some ideas that we would have liked to try during the description and the development of the neutrosophic topological space in the future work.

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